The Cofinality of the Saturated Uncountable Random Graph

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Abstract

Assuming CH, let Γ_{ω_1} be the saturated random graph of cardinality ω_1. In this paper we prove that it is consistent that $cf(Aut(\Gamma_{\omega_1}))$ and 2^{ω_1} can be any two prescribed regular cardinals subject only to the requirement $\omega_1 < cf(Aut(\Gamma_{\omega_1})) \leq 2^{\omega_1}$.

1 Introduction

Suppose that G is a group which is not finitely generated. Then G can be expressed as the union of a chain of proper subgroups. The cofinality of G, written $cf(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. In [**6**], Macpherson and Neumann proved that for any infinite cardinal κ, $cf(Sym(\kappa)) > \kappa$. In [**8**], Sharp and Thomas proved that it is consistent that $cf(Sym(\omega))$ and 2^ω can be any two prescribed regular, uncountable cardinals, subject only to the obvious requirement that $cf(Sym(\omega)) \leq 2^\omega$. Then, in [**9**], Sharp and Thomas considered the relationship between $cf(Sym(\omega))$ and two well-known cardinal invariants of the continuum, the dominating number $\mathfrak{d}$ and the bounding number $\mathfrak{b}$. They proved that $cf(Sym(\omega)) \leq \mathfrak{d}$, and that both $cf(Sym(\omega)) < \mathfrak{b}$ and $\mathfrak{b} < cf(Sym(\omega))$ are consistent with ZFC. In [**7**], Mildenberger and Shelah proved that $\mathfrak{g} < cf(Sym(\omega))$ is consistent, where $\mathfrak{g}$ is the groupwise density number.

If we regard $Sym(\kappa)$ as the automorphism group of $\langle \kappa; \rangle$, the "trivial structure of cardinality κ," then it is natural to try to compare $cf(Sym(\kappa))$ and $cf(Aut(\mathcal{M}))$, where $\mathcal{M}$ is another structure of cardinality κ. In [**10**], Thomas showed that if $\mathcal{M}$ is a countable, ω-categorical structure, then $cf(Aut(\mathcal{M})) \leq cf(Sym(\omega))$. I conjecture that an even stronger statement holds:

Conjecture. Let κ be a regular, infinite cardinal such that $\kappa^{<\kappa} = \kappa$, and $\mathcal{M}$ a saturated structure of cardinality κ. Then $cf(Aut(\mathcal{M})) \leq cf(Sym(\kappa))$.

There exist countable, ω-categorical structures $\mathcal{M}$ such that the inequality $cf(Aut(\mathcal{M})) < cf(Sym(\omega))$ holds. For example, in [**4**], Lascar showed that there exists a countable ω-categorical structure $\mathcal{B}$ such that the product of countably many cyclic groups of order 2 is a homomorphic image of $Aut(\mathcal{B})$. It follows that $cf(Aut(\mathcal{B})) = \omega$.

In [**10**], Thomas also showed that if $\mathcal{M}$ is a vector space of dimension ω over a finite field $\mathbb{F}$, then $cf(Aut(\mathcal{M})) = cf(Sym(\omega))$. On the other hand, the following question is open:

Question 1. Is it consistent that there exists a countable ω-categorical structure $\mathcal{M}$ such that

$$\omega < cf(Aut(\mathcal{M})) < cf(Sym(\omega))?$$

Let Γ be the countable random graph (see [**2**, pp. 37-38]), which is uniquely characterized up to isomorphism among countable graphs by the following property:

$(*)$ *If U, V are disjoint, finite sets of vertices in Γ, then there is a vertex x of Γ which is adjacent to all vertices in U and to no vertices in V.*

In [**3**], Hodges, Hodkinson, Lascar, and Shelah showed that $cf(Aut(\Gamma)) > \omega$. In [**12**], I proved that it is consistent that $cf(Aut(\Gamma))$ and 2^ω can be any two prescribed regular, uncountable cardinals, subject only to the obvious requirement that $cf(Aut(\Gamma)) \leq 2^\omega$.

In this paper, we generalize these results to models of $Th(\Gamma)$ in higher cardinalities. We work under CH and consider the saturated model of $Th(\Gamma)$ of cardinality $\omega_1 = 2^\omega$ which we denote by Γ_{ω_1}. Equivalently, Γ_{ω_1} is the unique graph of cardinality ω_1 satisfying the following property:

$(*)$*If U, V are disjoint, countable sets of vertices in Γ_{ω_1}, then there is a vertex x of Γ_{ω_1} which is adjacent to all vertices in U and to no vertices in V.*

It should be noted that we restrict ourselves to cardinality ω_1 for clarity of presentation only. Assuming $\kappa^{<\kappa} = \kappa$, we can consider the saturated model of $Th(\Gamma)$ of cardinality κ, and prove all of the analogous results for Γ_κ. The analogue of Question 1 in higher cardinalities is open as well.

Question 2. Is it consistent with CH that there exists a saturated structure $\mathcal{M}$ of cardinality ω_1 such that

$$\omega_1 < cf(Aut(\mathcal{M})) < cf(Sym(\omega_1))?$$

In particular, can we separate $cf(Sym(\omega_1))$ from $cf(Aut(\Gamma_{\omega_1}))$?

Question 3. Is it consistent that

$$cf(Aut(\Gamma_{\omega_1})) < cf(Sym(\omega_1))?$$

In section 3 we will prove the following result:

Theorem 1.1 Let Γ_{ω_1} be the saturated model of $Th(\Gamma)$ of cardinality ω_1 and let $G = Aut(\Gamma_{\omega_1})$. Then $cf(G) > \omega_1$.

In section 5 we will show that it is consistent that $cf(G)$ and 2^{ω_1} are any two regular cardinals subject only to the restrictions given by Theorem 1.1 and $cf(G) \leq 2^{\omega_1}$.

Theorem 1.2 Let Γ_{ω_1} be the saturated model of $Th(\Gamma)$ of cardinality ω_1 and let $G = Aut(\Gamma_{\omega_1})$. Let $M \models GCH$ and $\omega_2 \leq \kappa \leq \theta$ regular cardinals. Then there is a $\mathbb{P} \in M$ such that ($\mathbb{P}$ is ω_1-closed and $\mathbb{P}$ has $\omega_2 -$ c.c.$)^M$ and $M^{\mathbb{P}} \models CH + cf(G) = \kappa \leq \theta = 2^{\omega_1}$.

Theorems 1.1 and 1.2 are proved using generic sequences of automorphisms (as defined in [**5**] and [**11**]). The proof of Theorem 1.1 uses a binary tree argument similar to that used in [**5**]. The proof of Theorem 1.2 is similar to the proof of Theorem 1.2 in [**12**] although it is technically more complex. We will use a countable support iteration of ω_1-linked, countably compact notions of forcing (as defined in [**1**]).

For the remainder of this paper we work under CH and let $G = Aut(\Gamma_{\omega_1})$.

Definition 1.3 If $A \subseteq \Gamma_{\omega_1}$, then

$$G_A = \{g \in G \mid g(a) = a \text{ for all } a \in A\}$$

is the pointwise stabilizer of A.

Notation. We write $X \prec Y$ to mean "X is an elementary submodel of Y".

Definition 1.4. Let H be a subgroup of G. Then H is open in G if there exists a countable $\Gamma \prec \Gamma_{\omega_1}$ such that $G_\Gamma \leq H$.

Remark. H is open in G iff H is open in the topological space with basis

$$\{gG_A \mid g \in G, A \subseteq \Gamma_{\omega_1}, |A| < \omega_1\}.$$

The next theorem will be used in the proofs of Theorems 1.1 and 1.2.

Theorem 1.5 If H is an open subgroup of G, then there is $g \in G$ such that $H \cup \{g\}$ generates G.

Proof. Let $G_\Gamma \leq H$. Let Γ' be a countable random graph disjoint from Γ such that $\Gamma' \prec \Gamma_{\omega_1}$ and there are no edges between Γ' and Γ. Let $h : \Gamma \to \Gamma'$ be an isomorphism, and extend h to $g \in G$. We will show that $G = \langle G_\Gamma, g\rangle$.

Claim 1. $G_{\Gamma'} \leq \langle G_\Gamma, g\rangle$.

Proof. Clearly $G_{\Gamma'} = gG_\Gamma g^{-1}$. □

Claim 2. Let Γ'' be a countable random graph disjoint from Γ such that $\Gamma'' \prec \Gamma_{\omega_1}$ and there are no edges between Γ'' and Γ. Then $G_{\Gamma''}$ is contained in $\langle G_\Gamma, g\rangle$.

Proof. Clearly there exists $k \in G_\Gamma$ such that $k[\Gamma'] = \Gamma''$. Hence by Claim 1, $G_{\Gamma''} = kG_{\Gamma'}k^{-1} \leq \langle G_\Gamma, g \rangle$. □

Now, to finish the proof of Theorem 1.5, suppose that $k \in G$. Let Γ'' be a countable random graph disjoint from Γ and $k[\Gamma]$ such that $\Gamma'' \prec \Gamma_{\omega_1}$ and there are no edges between Γ'' and Γ and no edges between Γ'' and $k[\Gamma]$. Let $f \in G_{\Gamma''}$ be such that $s = fk \in G_\Gamma$. Then $k = f^{-1}s \in \langle G_\Gamma, g \rangle$. □

In particular, if $G = \cup_{\alpha<\lambda} H_\alpha$ is the union of a chain of proper subgroups, then no H_α is open, for $\alpha < \lambda$.

It will be convenient for us to use the following definition.

Definition 1.6 We define a *random graph* to be any model of $Th(\Gamma)$.

The results of this paper form part of the author's Phd thesis written under the supervision of Simon Thomas. The author would like to thank Simon Thomas for his enlightening discussions and encouragement.

2 Generic Sequences of Automorphisms

In this section, following Truss [**10**], we define the notion of a generic sequence of automorphisms of Γ_{ω_1}. The existence of generic sequences of length ω_1 was already proved in [**5**]. We repeat many of the results from [**5**] in this section as we will need them in the proof of Theorem 1.1. We then show that given a family of ω_1 subgroups of $G = Aut(\Gamma_{\omega_1})$ that are not open, we can find a generic sequence of automorphisms of length ω_1 such that uncountably many of the generics lie outside each given subgroup.

Notation. If Γ is a random graph and $g, h \in Aut(\Gamma)$ then the conjugate of g by h is $g^h = hgh^{-1}$.

Definition 2.1 Let Γ, Γ' and Γ'' be random graphs with $\Gamma \prec \Gamma' \prec \Gamma''$ and let $g = (g_\alpha)_{\alpha<\beta}$ be a sequence of automorphisms of Γ'.
(1) The *restriction* of g to Γ is defined to be $g \restriction \Gamma = (g_\alpha \restriction \Gamma)_{\alpha<\beta}$.
(2) If $f \in Aut(\Gamma')$, then the *conjugate* of g by f is defined to be $g^f = (g^f_\alpha)_{\alpha<\beta}$.

(3) A sequence of automorphisms h of Γ'' is an *extension* of g if $h \restriction \Gamma' = g$.

Remark. In Definition 2.1(1), $g \restriction \Gamma$ is not necessarily a sequence of automorphisms of Γ. However, whenever we use this notation we will have that $g_\alpha \restriction \Gamma \in Aut(\Gamma)$ for each $\alpha < \beta$.

Notation. Let $\mathcal{L}$ be the language of graphs together with β new 1-place function symbols $(\mathfrak{g}_\alpha)_{\alpha<\beta}$. If Γ is a random graph and $(g_\alpha)_{\alpha<\beta}$ is a sequence of automorphisms of Γ, then $(\Gamma, g_\alpha)_{\alpha<\beta}$ is the expansion of Γ to $\mathcal{L}$ where each $\mathfrak{g}_\alpha$ is interpreted by g_α.

Definition 2.2 Let Γ be a random graph, let β be an ordinal and let $\mathcal{L}$ be the expansion of the language of graphs to include β new 1-place function symbols as above.
(1) $\Delta(\Gamma) = \{\varphi(\overline{a}) \mid \overline{a}$ is a finite sequence from Γ and $\Gamma \models \varphi(\overline{a})\}$ is the *elementary diagram* of Γ. (Here φ is a first-order formula in the language of graphs.)
(2) V_β is the collection of $\mathcal{L}$-sentences expressing that each $\mathfrak{g}_\alpha$ is interpreted by an automorphism of Γ.
(3) If $g = (g_\alpha)_{\alpha<\beta}$ is a sequence of automorphisms of Γ then $T_0(g)$ is defined to be the following set of sentences in $\mathcal{L}$.

$$T_0(g) = \Delta(\Gamma) \cup V_\beta \cup \{\mathfrak{g}_\alpha(a) = b \mid \alpha < \beta,\ a, b \in \Gamma \text{ and } g_\alpha(a) = b\}.$$

(4) $T(g)$ is defined to be the set of finite conjunctions of formulae in $T_0(g)$.

Thus, a model of $T(g)$ is (isomorphic to) a random graph $\Gamma' \succ \Gamma$ together with a sequence of automorphisms $(h_\alpha)_{\alpha<\beta}$ extending g. Conversely, if g and h are sequences of automorphisms of Γ and Γ' respectively where $\Gamma \prec \Gamma'$ and h extends g, then $T(g) \subseteq T(h)$.

Definition 2.3 Let Γ be a random graph and $g = (g_\alpha)_{\alpha<\beta}$ a sequence of automorphisms of Γ. We say that g is *existentially closed* if, whenever $\Gamma' \succ \Gamma$, $h = (h_\alpha)_{\alpha<\beta}$ extends g and $\varphi(\overline{a}, \overline{b}) \in T(h)$ where $\overline{a}$ is a finite sequence from Γ and $\overline{b}$ is a finite sequence from Γ', there exists $\overline{c}$ in Γ such that $\varphi(\overline{a}, \overline{c}) \in T(g)$.

Remark. Let $g = (g_\alpha)_{\alpha<\beta}$ be an existentially closed sequence of automorphisms of Γ and let $f : \Gamma \to \Gamma'$ be an isomorphism. It follows easily from

Definition 2.3 that $g^f = (fg_\alpha f^{-1})_{\alpha<\beta}$ is an existentially closed sequence of the random graph Γ'.

Theorem 2.4 Let Γ, Γ' and Γ'' be countable random graphs such that $\Gamma \prec \Gamma'$ and $\Gamma \prec \Gamma''$. Let $g = (g_\alpha)_{\alpha<\beta}$ be an existentially closed sequence of automorphisms of Γ and $g^1 = (g^1_\alpha)_{\alpha<\beta}$ and $g^2 = (g^2_\alpha)_{\alpha<\beta}$ be two extensions of g over Γ' and Γ'' respectively. Then there exists a countable random graph $\Gamma''' \succ \Gamma$, an extension $g^3 = (g^3_\alpha)_{\alpha<\beta}$ of g over Γ''' and elementary embeddings f_1, f_2 from Γ', Γ'' respectively into Γ''' such that $f_1 \restriction \Gamma = f_2 \restriction \Gamma = id \restriction \Gamma$ and g^3 extends both $(g^1)^{f_1}$ and $(g^2)^{f_2}$.

Proof. See [**5**, Theorem 3]. □

The next lemma states that existentially closed automorphisms are closed under unions.

Lemma 2.5 ([**5**], Lemma 4) Let β be an ordinal, $(X, \leq)$ a linearly ordered set, and for each $x \in X$, Γ_x a countable random graph and $g^x = (g^x_\alpha)_{\alpha<\beta}$ an existentially closed sequence of automorphisms of Γ_x such that, for any $x, y \in X$ if $x \leq y$, then $\Gamma_x \prec \Gamma_y$ and the sequence g^y extends g^x. Then we have that $g = \cup_{x\in X} g^x = (\cup_{x\in X} g^x_\alpha)_{\alpha<\beta}$ is an existentially closed sequence of automorphisms of $\Gamma = \cup_{x\in X}\Gamma_x$.

Proof. The result follows easily from the definition of an existentially closed sequence of automorphisms. □

Definition 2.6 Let Γ, Γ' and Γ'' be random graphs with $\Gamma \prec \Gamma'$ and $\Gamma \prec \Gamma''$, and let $g^0 = (g^0_\alpha)_{\alpha<\beta}$, $g^1 = (g^1_\alpha)_{\alpha<\beta}$ and $g^2 = (g^2_\alpha)_{\alpha<\beta}$ be sequences of automorphisms of Γ, Γ' and Γ'' respectively, g^1 and g^2 extending g^0. Then we say that g^1 and g^2 are *compatible* over Γ if there exist $\Gamma''' \succ \Gamma$, elementary embeddings f_1, f_2 from Γ', Γ'' respectively into Γ''' and a family of automorphisms $g^3 = (g^3_\alpha)_{\alpha<\beta}$ of Γ''' such that $f_1 \restriction \Gamma = f_2 \restriction \Gamma = id \restriction \Gamma$ and g^3 extends both $(g^1)^{f_1}$ and $(g^2)^{f_2}$.

If $\Gamma' \cap \Gamma'' = \Gamma$, then g^1 and g^2 are compatible if and only if $T(g^1) \cup T(g^2)$ is consistent. Theorem 2.4 says that two extensions of an existentially closed sequence of automorphisms are compatible.

Definition 2.7
(1) Let $g = (g_n)_{n<\omega}$ be a sequence of automorphisms of Γ_{ω_1}. We say that g is *generic* if the following condition holds:

Whenever $\Gamma \prec \Gamma_{\omega_1}$ is a countable random graph such that $g_n[\Gamma] = \Gamma$ for all $n < \omega$, Γ' is a countable random graph such that $\Gamma \prec \Gamma'$ and $h = (h_n)_{n<\omega}$ is a sequence of automorphisms of Γ' extending $g \restriction \Gamma$, then either h and g are incompatible or there is an elementary embedding $f : \Gamma' \longrightarrow \Gamma_{\omega_1}$ such that $f \restriction \Gamma = id \restriction \Gamma$ and g extends h^f.
(2) Let β be an infinite ordinal and let $g = (g_\alpha)_{\alpha<\beta}$ be a sequence of automorphisms of Γ_{ω_1}. We say that g is *generic* if every ω-subsequence of g is generic.

Proposition 2.8 Let $g = (g_\alpha)_{\alpha<\beta}$ be a generic sequence of automorphisms of Γ_{ω_1}, let $\Gamma \prec \Gamma_{\omega_1}$ be a countable random graph such that $g_\alpha[\Gamma] = \Gamma$ for each $\alpha < \beta$ and suppose that $(\Gamma, g_\alpha \restriction \Gamma)_{\alpha<\beta}$ is an elementary substructure of $(\Gamma_{\omega_1}, g_\alpha)_{\alpha<\beta}$. Then $g \restriction \Gamma = (g_\alpha \restriction \Gamma)_{\alpha<\beta}$ is existentially closed.

Proof. See [**5**, Proposition 7]. □

Proposition 2.9 Let Γ be a countable random graph such that $\Gamma \prec \Gamma_{\omega_1}$ and let $g = (g_n)_{n<\omega}$ be an existentially closed sequence of automorphisms of Γ. Assume that $h = (h_n)_{n<\omega}$ and $k = (k_n)_{n<\omega}$ are two generic sequences of automorphisms of Γ_{ω_1} both extending g. Then there exists $f \in G_\Gamma$ such that $h = k^f$.

Proof. See [**5**, Proposition 8]. □

Lemma 2.10 Let $(g_{\alpha,\beta})_{\alpha,\beta<\omega_1}$ be a matrix of automorphisms of Γ_{ω_1} and let $(\Gamma_\alpha)_{\alpha<\omega_1}$ be a sequence of countable random graphs such that $\Gamma_\alpha \prec \Gamma_{\omega_1}$ for each $\alpha < \omega_1$. Then there exists a sequence $(h_\alpha)_{\alpha<\omega_1}$ such that

(i) $h_\alpha \in G_{\Gamma_\alpha}$ for each $\alpha < \omega_1$.
(ii) for each map $\delta : \omega_1 \longrightarrow \omega_1$, the sequence $(h_\alpha \cdot g_{\alpha,\delta(\alpha)})_{\alpha<\omega_1}$ is generic.

Proof. See [**5**,Lemma 9].

Proposition 2.11 Let $\lambda \leq \omega_1$ and let $\{H_\alpha \mid \alpha < \lambda\}$ be such that H_α is a subgroup of G which is not open for each $\alpha < \lambda$. Then there exists a generic sequence $\mathcal{F} = (g_\beta)_{\beta<\omega_1}$ such that for each $\alpha < \lambda$, each countable random graph $\Gamma \prec \Gamma_{\omega_1}$ and each $h \in Aut(\Gamma)$,

$$|\{\beta < \omega_1 \mid g_\beta \restriction \Gamma = h \wedge g_\beta \notin H_\alpha\}| = \omega_1.$$

Proof. Let

$$X = \{(\Gamma, f) \mid \Gamma \prec \Gamma_{\omega_1}, |\Gamma| = \omega, f \in Aut(\Gamma)\}.$$

Note that $|X| = \omega_1$. Let $((\Gamma_\xi, f_\xi))_{\xi<\omega_1}$ be a sequence of elements of X such that for all $(\Gamma, f) \in X$ the set $\{\xi < \omega_1 \mid (\Gamma_\xi, f_\xi) = (\Gamma, f)\}$ has cardinality ω_1. Let $K_\alpha = \omega_1 \times \{\alpha\}$ for each $\alpha < \lambda$ and let $I = \omega_1 \times \lambda = \cup_{\alpha<\lambda} K_\alpha$. Also, let $\Gamma_i = \Gamma_\xi$ and $f_i = f_\xi$ where $i = (\xi, \alpha)$.

We define $g_{i,\beta} \in G$ for each $i \in I$ and $\beta < \omega_1$ so that if $i \in K_\alpha$, then $g_{i,\beta} \restriction \Gamma_i = f_i$. Moreover, we demand that the set $\{g_{i,\beta}\}_{\beta<\omega_1}$ meets at least two cosets of H_α in G. This is possible because as H_α is not open, none of its cosets contains a non-empty open set.

Applying Lemma 2.10 we get a family $(h_i)_{i\in I}$ such that

(i) $h_i \in G_{\Gamma_i}$ for each $i \in I$.

(ii) $(h_i \cdot g_{i,\delta(i)})_{i\in I}$ is generic for each map $\delta : I \longrightarrow \omega_1$.

Choose a map $\delta : I \longrightarrow \omega_1$ so that if $i \in K_\alpha$ then $g_{i,\delta(i)} \notin h_i^{-1} H_\alpha$.

The sequence $(h_i \cdot g_{i,\delta(i)})_{i\in I}$ satisfies our requirements. □

3 Proof of Theorem 1.1

In this section we prove the first of the two results stated in the Introduction. We first show that if $G = \cup_{\alpha<\lambda} H_\alpha$ is the union of a chain of proper subgroups, then $\lambda \neq \omega_1$.

Theorem 3.1 Let Γ_{ω_1} be the saturated model of $Th(\Gamma)$ of cardinality ω_1 and let $G = Aut(\Gamma_{\omega_1})$. Then G cannot be expressed as the union of a chain of ω_1 proper subgroups.

Proof. Suppose that $G = \cup_{\alpha<\omega_1} H_\alpha$ is a chain of proper subgroups. By Theorem 1.5, each H_α is not open. Let $\mathcal{S} = 2^{<\omega_1}$ denote the set of sequences of 0 and 1 of length less than ω_1, and $\mathcal{S}^* = \{s \in \mathcal{S} \mid$ the length of s is a successor$\}$. By Proposition 2.11, there is a generic sequence $\mathcal{F} = (g_\beta)_{\beta<\omega_1}$ such that for each $\alpha < \omega_1$, each countable random graph $\Gamma \prec \Gamma_{\omega_1}$ and each $h \in Aut(\Gamma)$,

$$|\{\beta < \omega_1 \mid g_\beta \restriction \Gamma = h \wedge g_\beta \notin H_\alpha\}| = \omega_1.$$

Let $(a_\beta)_{\beta<\omega_1}$ be an enumeration of Γ_{ω_1}. We construct, by induction on $s \in \mathcal{S}$, a countable random graph $\Gamma_s \prec \Gamma_{\omega_1}$, an automorphism $f_s \in Aut(\Gamma_s)$, an ordinal β_s, and, if $s \in \mathcal{S}^*$, automorphisms h_s and k_s in G_{Γ_s} so that the following conditions hold:

(1) the maps $s \to \Gamma_s$ and $s \to f_s$ are increasing and continuous.

(2) If $length(s) = \xi$, then $\xi \le \beta_s < \omega_1$.

(3) for all $\xi < \omega_1$, for all $s \in 2^\xi$, $h_{s^\wedge 0} \in H_{\beta_s}$ and $h_{s^\wedge 1} \notin H_{\beta_s}$.

(4) for all $s \in \mathcal{S}$, $k_{s^\wedge 0} = k_{s^\wedge 1}$.

(5) for all $s \in \mathcal{S}$, for all $t \in \mathcal{S}^*$ such that $t \le s$, $h_t[\Gamma_s] = \Gamma_s$ and $(h_t \restriction \Gamma_s \mid t \le s \text{ and } t \in \mathcal{S}^*)$ is existentially closed.

(6) for all $s \in \mathcal{S}$, for all $t \in \mathcal{S}^*$ such that $t \le s$, $(h_t \restriction \Gamma_s)^{f_s} = k_t \restriction \Gamma_s$.

(7) for all $s \in \mathcal{S}$ and $\beta < length(s)$, $a_\beta \in \Gamma_s$.

(8) for all $s \in \mathcal{S}$, the families $(h_t \mid t \le s, t \in \mathcal{S}^*)$ and $(k_t \mid t \le s, t \in \mathcal{S}^*)$ are sequences of distinct elements of $\mathcal{F}$.

If $s = \emptyset$, let Γ_s be any countable random graph such that $\Gamma_s \prec \Gamma_{\omega_1}$ and let $f_s = id \restriction \Gamma_s$. If $length(s)$ is a limit, let $\Gamma_s = \cup_{t<s}\Gamma_t$ and $f_s = \cup_{t<s} f_t$.

Assume that β_t has been defined for all $t < s$, Γ_s, f_s have been defined for all $t \le s$ and h_s, k_s have been defined for all $t \in \mathcal{S}^*$ with $t \le s$. We show how to define β_s, $\Gamma_{s^\wedge 0}$, $\Gamma_{s^\wedge 1}$, $f_{s^\wedge 0}$, $f_{s^\wedge 1}$,$h_{s^\wedge 0}$, $h_{s^\wedge 1}$, $k_{s^\wedge 0}$ and $k_{s^\wedge 1}$. First, let $h_{s^\wedge 0}$ be an element in $\mathcal{F}$ not in $\{h_t \mid t \in \mathcal{S}^*, t \le s\}$.

Extend f_s to $f \in G$ in such a way that $(h_t)^f = k_t$ for all $t \in \mathcal{S}^*$, $t \le s$. To do this, first extend f_s arbitrarily to $f_1 \in G$. Then the families $((h_t)^{f_1} \mid t \le s, t \in \mathcal{S}^*)$ and $(k_t \mid t \le s, t \in \mathcal{S}^*)$ are generic, and they agree on Γ_s. Also, $((h_t)^{f_1} \restriction \Gamma_s \mid t \le s, t \in \mathcal{S}^*)$ is existentially closed, so, by Proposition 2.9, there exists $f_2 \in G_{\Gamma_s}$ such that for all $t \le s$, $t \in \mathcal{S}^*$, $k_t = (h_t)^{f_2 f_1}$. Take $f = f_2 \cdot f_1$.

Now, choose $\Gamma_{s^\wedge 0} = \Gamma_{s^\wedge 1}$ in such a way that:

(a) it is closed for h_t, for $t \le s^\wedge 0$ and for f.

(b) it contains Γ_s and a_α, where $\alpha = length(s)$.

(c) $(\Gamma_{s^\wedge 0}, h_t \restriction \Gamma_{s^\wedge 0})_{t \le s^\wedge 0} \prec (\Gamma_{\omega_1}, h_t)_{t \le s^\wedge 0}$.

Set $f_{s^\wedge 0} = f_{s^\wedge 1} = f \restriction \Gamma_{s^\wedge 0}$. Suppose that $s \in 2^\xi$. Let β_s be the least ordinal β such that $\beta \ge \xi$ and $h_{s^\wedge 0} \in H_\beta$. Let $h_{s^\wedge 1}$ be an element of $\mathcal{F}$ extending $h_{s^\wedge 0} \restriction \Gamma_{s^\wedge 0}$ not in H_{β_s} and not in $\{h_t \mid t \le s, t \in \mathcal{S}^*\}$, and $k_{s^\wedge 0} = k_{s^\wedge 1}$ an element of $\mathcal{F}$ extending $(h_{s^\wedge 0} \restriction \Gamma_{s^\wedge 0})^f$ which is not in $\{k_t \mid t \le s, t \in \mathcal{S}^*\}$.

We are now able to reach a contradiction: for each $\sigma \in 2^{\omega_1}$, let $f_\sigma = \cup_{s<\sigma} f_s$. Then $f_\sigma \in G$ and for all $t < \sigma$, $t \in \mathcal{S}^*$, $h_t^{f_\sigma} = k_t$.

Since $G = \cup_{\alpha<\omega_1} H_\alpha$, there are $\xi < \omega_1$ and $\Sigma \subseteq 2^{\omega_1}$ such that $|\Sigma| = \omega_2$ and $f_\sigma \in H_\xi$ for all $\sigma \in \Sigma$. Choose $\gamma \geq \xi$, and $\sigma, \tau \in \Sigma$ such that $length(s) = \gamma$, $s^\wedge 0 < \sigma$ and $s^\wedge 1 < \tau$. Then $f_\sigma h_{s^\wedge 0} f_\sigma^{-1} = k_{s^\wedge 0} = k_{s^\wedge 1} = f_\tau h_{s^\wedge 1} f_\tau^{-1}$. So, $h_{s^\wedge 0} = f_\sigma^{-1} f_\tau h_{s^\wedge 1} f_\tau^{-1} f_\sigma$. Since $h_{s^\wedge 0} \in H_{\beta_s}$ and $h_{s^\wedge 1} \notin H_{\beta_s}$, we have that $f_\sigma^{-1} \cdot f_\tau \notin H_{\beta_s} \supseteq H_\xi$, a contradiction. □

We now complete the proof of Theorem 1.1 by showing that if $G = \cup_{\alpha<\lambda} H_\alpha$ is the union of a chain of proper subgroups, then $\lambda \neq \omega$.

Theorem 1.1 Let Γ_{ω_1} be the saturated model of $Th(\Gamma)$ of cardinality ω_1 and let $G = Aut(\Gamma_{\omega_1})$. Then $cf(G) > \omega_1$.

Proof. Suppose that $G = \cup_{n<\omega} H_n$ is a chain of proper subgroups. By Theorem 1.5, each H_n is not open. Let $\mathcal{S} = 2^{<\omega}$ denote the set of sequences of 0 and 1 of length less than ω. By Proposition 2.11, there is a generic sequence $\mathcal{F} = (g_\beta)_{\beta<\omega_1}$ such that for each $n < \omega$, each countable random graph $\Gamma \prec \Gamma_{\omega_1}$ and each $h \in Aut(\Gamma)$,

$$|\{\beta < \omega_1 \mid g_\beta \restriction \Gamma = h \wedge g_\beta \notin H_n\}| = \omega_1.$$

As in the proof of Theorem 3.1, we construct, by induction on $s \in \mathcal{S}$, a countable random graph $\Gamma_s \prec \Gamma_{\omega_1}$, an automorphism $f_s \in Aut(\Gamma_s)$, a natural number n_s and, if $s \in \mathcal{S} \setminus \{\emptyset\}$, automorphisms h_s and k_s in G_{Γ_s} so that the following conditions hold:

(1) the maps $s \to \Gamma_s$ and $s \to f_s$ are increasing and continuous.

(2) If $length(s) = n$, then $n_s \geq n$.

(3) for all $n < \omega$, for all $s \in 2^n$, $h_{s^\wedge 0} \in H_{n_s}$ and $h_{s^\wedge 1} \notin H_{n_s}$.

(4) for all $s \in \mathcal{S} \setminus \{\emptyset\}$, $k_{s^\wedge 0} = k_{s^\wedge 1}$.

(5) for all $s \in \mathcal{S} \setminus \{\emptyset\}$, for all $\emptyset \neq t \leq s$, $h_t[\Gamma_s] = \Gamma_s$ and $(h_t \restriction \Gamma_s \mid \emptyset \neq t \leq s)$ is existentially closed.

(6) for all $s \in \mathcal{S} \setminus \{\emptyset\}$, for all $\emptyset \neq t \leq s$, $(h_t \restriction \Gamma_s)^{f_s} = k_t \restriction \Gamma_s$.

(7) for all $s \in \mathcal{S}$, the families $(h_t \mid \emptyset \neq t \leq s)$ and $(k_t \mid \emptyset \neq t \leq s)$ are sequences of distinct elements of $\mathcal{F}$.

For each $\sigma \in 2^\omega$, let $\Gamma_\sigma = \cup_{s<\sigma} \Gamma_s$, and let $f_\sigma = \cup_{s<\sigma} f_s \in Aut(\Gamma_s)$. Then for all $t < \sigma$, $t \neq \emptyset$, $(h_t \restriction \Gamma_\sigma)^{f_\sigma} = k_t \restriction \Gamma_\sigma$. Extend each f_σ to $\pi_\sigma \in G$ such that for all $\emptyset \neq t < \sigma$, $h_t^{\pi_\sigma} = k_t$.

Clearly there are ω_1 branches σ and a fixed integer $m < \omega$ such that $\pi_\sigma \in H_m$. So we can find $\sigma \neq \tau$ such that $n = \min\{l \mid \sigma \restriction l \neq \tau \restriction l\} > m$.

Now, by construction, there is $l \geq n > m$ such that $h_{\sigma\restriction n} \in H_l$ if and only if $h_{\tau\restriction n} \notin H_l$. But $\pi_\sigma h_{\sigma\restriction n}\pi_\sigma^{-1} = \pi_\tau h_{\tau\restriction n}\pi_\tau^{-1}$, a contradiction as before.

□

4 Countable Support Iterations

To prove Theorem 1.2 we will need to use a countable support iteration of the form $\langle\langle \mathbb{P}_\xi \mid \xi \leq \kappa\rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \kappa\rangle\rangle$. We will ensure in our construction that $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is countably compact and ω_1-linked and $|\dot{\mathbb{Q}}_\xi| \leq \kappa$" for each $\xi < \kappa$. By results presented in [**1**] this will ensure that all cardinals are preserved in this extension. In this section we will repeat the relevant definitions and results from [**1**] . We also show that $Fn(2^{\omega_1}, \omega_1, \omega_1)$ is ω_1-linked.

Theorem 4.1 Let δ be an infinite ordinal and $\langle\langle \mathbb{P}_\xi \mid \xi \leq \delta\rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \delta\rangle\rangle$ be a countable support iteration such that $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is countably closed" for each $\xi < \delta$. Then $\mathbb{P}_\delta$ is countably closed.

Proof. See [**1**, Theorem 2.5]. □

Definition 4.2
(1) We say that a partial order $\mathbb{P}$ is ω_1-*linked* if there exists a mapping $f : \mathbb{P} \to \omega_1$ such that for each $\alpha < \omega_1$, $f^{-1}\{\alpha\}$ is pairwise compatible. f is called a *linking function* for $\mathbb{P}$.
(2) We say that a partial order $\mathbb{P}$ is *countably compact* if for any countable set $A \subseteq \mathbb{P}$ the following condition holds:

(*) if for any finite set $F \subseteq A$ there is $p \in \mathbb{P}$ such that for all $q \in F$, $p \leq q$, then there is $p \in \mathbb{P}$ such that for all $q \in A$, $p \leq q$.

Remarks.
(1) If $\mathbb{P}$ is ω_1-linked, then $\mathbb{P}$ has the ω_2-c.c.
(2) If $\mathbb{P}$ is countably compact, then $\mathbb{P}$ is countably closed.

Definition 4.3 $Fn(I, J, \lambda)$ is the partial ordering

$$\{p \mid |p| < \lambda,\ p \text{ is a function such that } \operatorname{dom} p \subseteq I \text{ and } \operatorname{ran} p \subseteq J\}.$$

$Fn(I, J, \lambda)$ is ordered by $p \leq q$ iff $p \supseteq q$.

Lemma 4.4 (CH) $Fn(2^{\omega_1}, \omega_1, \omega_1)$ is ω_1-linked.

Proof. For each $s \in [\omega_1]^\omega$, each collection $(\varphi_n)_{n<\omega}$ such that $\varphi_n : s \to 2$ for each $n < \omega$ and $n \neq m \to \varphi_n \neq \varphi_m$, and each $\Phi : \{\varphi_n\}_{n<\omega} \to \omega_1$, let

$$C(s, (\varphi_n)_{n<\omega}, \Phi) = \{p \in Fn(2^{\omega_1}, \omega_1, \omega_1) \mid \text{dom } p = \{f_n\}_{n<\omega},\ f_n \restriction s = \varphi_n \\ \text{for each } n \text{ and } p(f_n) = \Phi(\varphi_n) \text{ for each } n\}.$$

It is easy to see that these sets cover $Fn(2^{\omega_1}, \omega_1, \omega_1)$, each of these sets is pairwise compatible and, using CH, there are ω_1 of them. □

Theorem 4.5 (CH) Let δ be an ordinal and let $\langle\langle \mathbb{P}_\xi \mid \xi \leq \delta\rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \delta\rangle\rangle$ be a countable support iteration such that

$$\Vdash_{\mathbb{P}_\xi} \text{“}\dot{\mathbb{Q}}_\xi \text{ is countably compact and } \omega_1\text{-linked”}$$

for each $\xi < \delta$. Then $\mathbb{P}_\delta$ has the ω_2-c.c.

Proof. By the proof of Theorem 4.2 in [**1**] □

Theorem 4.6 Let θ be a cardinal such that $\theta^{\omega_1} = \theta$. Suppose that $\mathbb{P}$ has the ω_2-c.c, $|\mathbb{P}| \leq \theta$, and $\Vdash_{\mathbb{P}} |\dot{\mathbb{Q}}| \leq \theta$. Then $|\mathbb{P} * \dot{\mathbb{Q}}| \leq \theta$.

Remark. We identify elements $(p_1, \dot{q}_1)$, $(p_2, \dot{q}_2) \in \mathbb{P} * \dot{\mathbb{Q}}$ whenever we have that $(p_1, \dot{q}_1) \leq (p_2, \dot{q}_2) \leq (p_1, \dot{q}_1)$.

Proof. See [**1**, Lemma 3.2]. □

5 Proof of Theorem 1.2

We are now ready to begin the proof of Theorem 1.2.

Theorem 1.2 Let Γ_{ω_1} be the saturated model of $Th(\Gamma)$ of cardinality ω_1 and let $G = Aut(\Gamma_{\omega_1})$. Let $M \models GCH$ and $\omega_2 \leq \kappa \leq \theta$ regular cardinals. Then there is a $\mathbb{P} \in M$ such that $(\mathbb{P} \text{ is } \omega_1\text{−closed and } \mathbb{P} \text{ has } \omega_2 - \text{c.c.})^M$ and $M^{\mathbb{P}} \models CH + cf(G) = \kappa \leq \theta = 2^{\omega_1}$.

We will need to use generic sequences to ensure that certain density conditions are satisfied. Unfortunately, Definition 2.7 does not suffice for our purposes and we shall need to use the following slightly stronger version of a generic sequence.

Definition 5.1
(1) Let $g = (g_n)_{n<\omega}$ be a sequence of automorphisms of Γ_{ω_1}. We say that g is *strongly generic* if the following condition holds:

Whenever $\Gamma \prec \Gamma_{\omega_1}$ is a countable random graph with $g_n[\Gamma] = \Gamma$ for all $n < \omega$, Γ' is a countable random graph such that $\Gamma \prec \Gamma'$, $h = (h_n)_{n<\omega}$ is a sequence of automorphisms of Γ' extending $g \restriction \Gamma$, and C is a countable set such that $C \subseteq \Gamma_{\omega_1} \setminus \Gamma'$, then either h and g are incompatible or there is an elementary embedding $f : \Gamma' \longrightarrow \Gamma_{\omega_1}$ such that $f \restriction \Gamma = id \restriction \Gamma$, $f[\Gamma'] \cap C = \emptyset$, there are no edges between $f[\Gamma']$ and C, and g extends h^f.

(2) Let β be an infinite ordinal and let $g = (g_\alpha)_{\alpha<\beta}$ be a sequence of automorphisms of Γ_{ω_1}. We say that g is *strongly generic* if every ω-subsequence of g is strongly generic.

We now define the partial orders that will be needed in our construction.

Definition 5.2 Let $\rho : \Gamma_{\omega_1} \to \Gamma_{\omega_1}$ be a countable partial isomorphism. Then $\mathbb{Q}_\rho$ is the partial order consisting of all conditions φ such that:

(1) $\varphi \in Aut(\Gamma)$ for some countable random graph $\Gamma \prec \Gamma_{\omega_1}$.

(2) $\varphi \supseteq \rho$.

$\mathbb{Q}_\rho$ is ordered by $\varphi_1 \leq \varphi_2$ iff $\varphi_1 \supseteq \varphi_2$.

It is easy to see that $\mathbb{Q}_\rho$ is countably compact. Also, under CH we have that $|\mathbb{Q}_\rho| = 2^\omega = \omega_1$, and thus $\mathbb{Q}_\rho$ is ω_1-linked.

Definition 5.3 Let $\lambda \leq 2^{\omega_1}$ be a regular cardinal, $g = (g_\alpha \mid \alpha < \lambda)$ a sequence of elements of G, $S : \lambda \to [\lambda]^{\omega_1}$ and $S(\alpha) = S_\alpha$ for each $\alpha < \lambda$. Then $\mathbb{Q}_{g,S}$ is the partial order consisting of all conditions $\langle \varphi, F \rangle$ such that:

(1) $\varphi \in Aut(\Gamma)$ for some countable random graph $\Gamma \prec \Gamma_{\omega_1}$.

(2) $F : \lambda \to \lambda$ is a countable injective partial map.

(3) For all $\alpha \in \text{dom } F$, $F(\alpha) \in S_\alpha$.

(4) For all $\alpha \in \text{dom } F$, dom φ is invariant under g_α.

(5) $(\varphi g_{\alpha_n} \varphi^{-1} \restriction \varphi[\Gamma])_{n<\omega}$ is existentially closed, where dom $F = \{\alpha_n\}_{n<\omega}$.

(6) For all $\alpha \in \text{dom } F$ and $a \in \text{ran } \varphi$, $\varphi g_\alpha \varphi^{-1}(a) = g_{F(\alpha)}(a)$.

$\mathbb{Q}_{g,S}$ is ordered by $\langle \varphi_1, F_1 \rangle \leq \langle \varphi_2, F_2 \rangle$ iff $\varphi_1 \supseteq \varphi_2$ and $F_1 \supseteq F_2$.

It is easily checked that $\mathbb{Q}_{g,S}$ is countably compact. We will verify that $\mathbb{Q}_{g,S}$ is ω_1-linked as follows. Let $\mathbb{R} = Fn(\lambda, \omega_1, \omega_1)$ and let

$$\mathbb{T} = \{F \mid \text{for some } \varphi,\ \langle \varphi, F\rangle \in \mathbb{Q}_{g,S}\}.$$

By Lemma 4.4, $\mathbb{R}$ is ω_1-linked. Let f be a linking function for $\mathbb{R}$. Define a map $J : \mathbb{T} \to \mathbb{R}$ as follows. For each α, suppose that $S_\alpha = \{\alpha_\xi\}_{\xi<\omega_1}$. Let $F \in \mathbb{T}$. If $F(\alpha) = \alpha_\xi$, let $J(F)(\alpha) = \xi$. Then J is injective and preserves compatibility. Now, enumerate the countable partial isomorphisms of Γ_{ω_1} as $\{\varphi_\alpha\}_{\alpha<\omega_1}$. Define a map $h : \mathbb{P} \to \omega_1 \times \omega_1$ by $h(\langle \varphi_\alpha, F\rangle) = \langle \alpha, f(J(F))\rangle$. Let $k : \omega_1 \times \omega_1 \to \omega_1$ be a bijection. Then $k \circ h$ is a linking function for $\mathbb{P}$.

The partial order $\mathbb{Q}_\rho$ extends a strongly generic sequence of automorphisms in the following sense.

Lemma 5.4 Let $M \models ZFC$, and let $(g_\beta \mid \beta < \alpha)$ be a (possibly empty) strongly generic sequence of elements of G. Let $\rho : \Gamma_{\omega_1} \to \Gamma_{\omega_1}$ be a countable partial isomorphism, let $\mathbb{Q} = \mathbb{Q}_\rho$ and let $k \in M^{\mathbb{Q}}$ be the $\mathbb{Q}$-generic automorphism. Then for all $h \in G^M$, $(g_\beta \mid \beta < \alpha)^\wedge hk$ is strongly generic.

Proof. Let $(g_n)_{n<\omega}$ be a countable subset of $(g_\beta)_{\beta<\alpha}$. We show that the sequence $(g_n \mid n < \omega)^\wedge hk$ is strongly generic.

Let $\Gamma \prec \Gamma_{\omega_1}$ be a countable random graph with $g_n[\Gamma] = \Gamma$ for all $n < \omega$ and $hk[\Gamma] = \Gamma$, let Γ' be a countable random graph such that $\Gamma \prec \Gamma'$, let $\theta = (q_n)_{n<\omega}$ be a sequence of automorphisms of Γ' such that q_n extends $g_n \restriction \Gamma$ for each $n < \omega$ and let q be an automorphism of Γ' such that q extends $hk \restriction \Gamma$. Let C be a countable set such that $C \subseteq \Gamma_{\omega_1} \setminus \Gamma'$ and suppose that $(q_n \mid n < \omega)^\wedge q$ and $(g_n \mid n < \omega)^\wedge hk$ are compatible over Γ. It is enough to show that

$$\begin{aligned} D_{\Gamma,\Gamma'} = \{\varphi \in \mathbb{Q} \mid\ & \Gamma \subseteq \text{dom } \varphi,\ \text{there exists an elementary embedding} \\ & f : \Gamma' \to \Gamma_{\omega_1} \text{ such that } f \restriction \Gamma = id \restriction \Gamma, f[\Gamma'] \cap C = \emptyset, \text{ there are} \\ & \text{no edges between } f[\Gamma'] \text{ and } C,\ g_n \text{ extends } q_n^f \text{ for each } n < \omega \\ & \text{and } h\varphi \text{ extends } q^f\} \end{aligned}$$

is dense in $\mathbb{Q}$. Let $\varphi \in \mathbb{Q}$ be arbitrary. Without loss of generality, we may assume that $\Gamma \subseteq \text{dom } \varphi$ and $\Gamma_{\omega_1} \cap \Gamma' = \Gamma$. We let $f : \Gamma' \to \Gamma_{\omega_1}$ be an elementary embedding such that $(g_n)_{n<\omega}$ extends θ^f, $f \restriction \Gamma = id \restriction \Gamma$, $f[\Gamma'] \cap (C \cup \text{dom } \varphi) = \Gamma$ and there are no edges between $f[\Gamma']$ and $C \cup (\text{dom } \varphi \setminus \Gamma)$.

Let φ' be an extension of $\varphi \cup h^{-1}q^f$ to an automorphism of some countable random graph Γ'' containing dom $\varphi \cup f[\Gamma']$. Then $\varphi' \in \mathbb{Q}$ and $\varphi' \leq \varphi$. $\square$

Let ϵ be the ordinal product $\theta \cdot \kappa$. We construct in M, a countable support iteration of the form $\langle \langle \mathbb{P}_\xi \mid \xi \leq \epsilon \rangle, \langle \dot{\mathbb{Q}}_\xi \mid \xi < \epsilon \rangle \rangle$. We ensure in our construction that for each $\xi < \epsilon$,

(i) $\Vdash_{\mathbb{P}_\xi}$ "$\dot{\mathbb{Q}}_\xi$ is countably compact and ω_1-linked."
(ii) $\Vdash_{\mathbb{P}_\xi} |\dot{\mathbb{Q}}_\xi| \leq \theta$.

By (i), Theorem 4.1 and Theorem 4.5, for each $\xi \leq \epsilon$, $\mathbb{P}_\xi$ will be countably closed and ω_2-c.c. in M.
By (ii), Lemma 4.6 and induction on ξ, we have in M, $|\mathbb{P}_\xi| \leq \theta$ for each $\xi \leq \epsilon$.

Definition 5.5
(1) $Z = [\Gamma_{\omega_1} \times \Gamma_{\omega_1}]^\omega$.
(2) For each regular $\lambda \leq 2^{\omega_1}$, let $Y_\lambda = [\Gamma_{\omega_1} \times \Gamma_{\omega_1}]^\omega \times [\lambda \times \lambda]^\omega$.

Remarks.
(1) Each partial order of the form $\mathbb{Q}_\rho$ is a subset of Z.
(2) Each partial order of the form $\mathbb{Q}_{g,S}$ is a subset of Y_λ for some regular $\lambda \leq 2^{\omega_1}$.
(3) $|Z| = (\omega_1^\omega \cdot \omega_1^\omega) \cdot \omega_1^\omega = \omega_1$.
(4) $|Y_\lambda| = (\omega_1^\omega \cdot \omega_1^\omega) \cdot (\lambda^\omega \cdot \lambda^\omega) = \lambda$ (here we use the facts that $cf(\lambda) > \omega$ and GCH holds in the ground model).

Since $\mathbb{P}_\epsilon$ is countably closed, whenever $\mathcal{G}$ is $\mathbb{P}_\epsilon$-generic over M, $M[\mathcal{G}] \models CH$. By (ii), in M there are at most $(\theta^{\omega_1})^{\omega_1} = \theta$ nice $\mathbb{P}_\epsilon$-names for subsets of $\check{\omega}_1$. So, whenever $\mathcal{G}$ is $\mathbb{P}_\epsilon$-generic over M, $M[\mathcal{G}] \models 2^{\omega_1} \leq \theta$. Also by (ii), for each $\xi < \epsilon$ and each regular $\lambda < \kappa$ there are at most $(\theta^{\omega_1})^{|Z|} = \theta$ nice $\mathbb{P}_\xi$-names for subsets of $\check{Z}$ and at most $(\theta^{\omega_1})^{|Y_\lambda|} = \theta$ nice $\mathbb{P}_\xi$-names for subsets of $\check{Y}_\lambda$. These remarks allow us to make the following construction:

Fix a mapping $\pi : \theta \longrightarrow \theta \times \theta$ such that for every $(\beta, \gamma) \in \theta \times \theta$, there are arbitrarily large $\xi < \theta$ such that $\pi(\xi) = (\beta, \gamma)$, and whenever $\pi(\xi) = (\beta, \gamma)$, then $\beta \leq \xi$. Suppose $\mathbb{P}_\xi$ has been determined. Let $\langle \dot{\mathbb{Q}}_\xi^\gamma \mid \gamma < \theta \rangle$ be an enumeration of $\mathbb{P}_\xi$-names such that

$\Vdash_{\mathbb{P}_\xi}$ "$\langle \dot{\mathbb{Q}}_\xi^\gamma \mid \gamma < \theta \rangle$ enumerates all partial orderings which are subsets of $\check{Z}$, or $\check{Y}_\lambda$ for some regular $\lambda < \kappa$."

Let $\pi(\xi) = (\beta, \gamma)$. Let $\dot{\mathbb{Q}}_\xi$ be the term which denotes the same partial ordering $\mathbb{Q}$ in $V[G_\xi]$ that $\dot{\mathbb{Q}}^\gamma_\beta$ denotes in $V[G_\beta]$ provided $\mathbb{Q}$ satisfies one of the following:
(i) $\mathbb{Q} = \mathbb{Q}_\rho$ for some countable partial isomorphism $\rho : \Gamma_{\omega_1} \to \Gamma_{\omega_1}$ or
(ii) there is a regular $\lambda < \min\{\kappa, (2^{\omega_1})^+\}$, a sequence $g = (g_\alpha \mid \alpha < \lambda)$ and a map $S : \lambda \to [\lambda]^{\omega_1}$ such that $\mathbb{Q} = \mathbb{Q}_{g,S}$ (Here $(2^{\omega_1})^+$ is computed in $V[G_\beta]$).

If $\dot{\mathbb{Q}}^\gamma_\beta$ does not satisfy (i) or (ii) in $V[G_\xi]$, then let $\dot{\mathbb{Q}}_\xi$ denote the trivial partial ordering. Note that

$$\Vdash_{\mathbb{P}_\xi} \text{"}\dot{\mathbb{Q}}_\xi \text{ is countably compact and } \omega_1\text{-linked, and } |\dot{\mathbb{Q}}_\xi| \leq \theta.\text{"}$$

Set $\mathbb{P} = \mathbb{P}_\epsilon$, and let $\mathcal{G}$ be $\mathbb{P}$-generic over M. Since $\mathbb{P}$ is ω_1-closed and ω_2-c.c. all cardinals are preserved. So, $M[\mathcal{G}] \models 2^{\omega_1} \geq \theta$, and thus $M[\mathcal{G}] \models 2^{\omega_1} = \theta$. We show that $M[\mathcal{G}] \models cf(G) = \kappa$.

Let $G_\alpha = \{g \in G \mid g \in M^{\mathbb{P}_\alpha}\}$ for each $\alpha < \epsilon$, and let $f : \kappa \to \epsilon$ be a cofinal map. Then $G^{M[\mathcal{G}]} = \cup_{\alpha<\kappa} G_{f(\alpha)}$. So, $M[\mathcal{G}] \models cf(G) \leq \kappa$.

To prove the other direction, suppose that $M[\mathcal{G}] \models G = \cup_{\alpha<\lambda} H_\alpha$ where $\lambda < \kappa$ is a regular cardinal. Then by Theorem 1.1, $\lambda > \omega_1$. By Theorem 1.5 each H_α is not open. We define a sequence $(g_\alpha)_{\alpha<\lambda}$ of elements of G and a strictly increasing sequence of ordinals $(\xi_\alpha)_{\alpha<\lambda}$ in $M[\mathcal{G}]$ so that:

(i) For each $\alpha < \lambda$, $g_\alpha \in H_{\xi_\alpha} \setminus \cup_{\beta<\xi_\alpha} H_\beta$.
(ii) For each $\alpha < \lambda$, $(g_\beta)_{\beta<\alpha}$ is strongly generic.
(iii) Suppose that $\gamma \leq \alpha < \lambda$ and that $\varphi \in Aut(\Gamma)$ for some countable random graph $\Gamma \prec \Gamma_{\omega_1}$ such that dom φ is invariant under g_γ. Then there exists $\beta > \alpha$ such that $\varphi g_\gamma \varphi^{-1}(a) = g_\beta(a)$ for all $a \in$ ran φ.

Choose $\xi < \epsilon$ such that $\mathbb{Q}_\emptyset$ is the denotation of $\dot{\mathbb{Q}}_\xi$. We can find a filter $\mathcal{H}$ in $M^{\mathbb{P}_{\xi+1}}$ which is $\mathbb{Q}_\emptyset$-generic over $M^{\mathbb{P}_\xi}$. Let $g_0 = \bigcup \mathcal{H}$. By Lemma 5.4, g_0 is a strongly generic element of G.

Let ξ_0 be the least ordinal such that $g_0 \in H_{\xi_0}$. Then g_0, ξ_0 satisfy (i) and (ii).

Now suppose that $(g_\beta)_{\beta<\alpha}$ has been defined satisfying (i) and (ii). Let J be the set of all pairs $\langle \gamma, \varphi \rangle$ satisfying the following conditions:

(a) $\gamma < \alpha$.
(b) $\varphi \in Aut(\Gamma)$ for some countable random graph $\Gamma \prec \Gamma_{\omega_1}$.
(c) dom φ is invariant under g_γ.

Then J has cardinality $\mu < \lambda$. Enumerate this set as $(\langle\gamma_\beta, \varphi_\beta\rangle)_{\alpha\leq\beta<\alpha+\mu}$, and for each β, let $h_\beta = \varphi_\beta g_{\gamma_\beta}\varphi_\beta^{-1} \restriction \text{ran } \varphi_\beta$. We extend each $h_{\alpha+\delta}$ to $g_{\alpha+\delta}$ satisfying (i) and (ii) as follows:

Suppose $(g_\beta)_{\beta<\alpha+\delta}$ and $(\xi_\beta)_{\beta<\alpha+\delta}$ satisfy (i) and (ii). Let $\rho = h_{\alpha+\delta}$ and $H = H_\mu$ where $\mu = sup_{\beta<\alpha+\delta}\xi_\beta < \lambda$. Since H is not open, we can choose $m \in G \setminus H$ such that m extends $h_{\alpha+\delta}$. There is $\xi < \epsilon$ such that $\mathbb{Q}_\rho$ is the denotation of $\dot{\mathbb{Q}}_\xi$, and we have m, $(g_\beta)_{\beta<\alpha+\delta} \in M^{\mathbb{P}_\xi}$. We can find a filter $\mathcal{H}$ in $M^{\mathbb{P}_{\xi+1}}$ which is $\mathbb{Q}_\rho$-generic over $M^{\mathbb{P}_\xi}$. Let $k = \bigcup\mathcal{H}$. Clearly, $k \in G$. By Lemma 5.4, in $M^{\mathbb{P}_\epsilon}$ $(g_\beta \mid \beta < \alpha + \delta)^\wedge k$ and $(g_\beta \mid \beta < \alpha + \delta)^\wedge mk$ are both strongly generic, and clearly $k \notin H$ or $mk \notin H$. Let $g_{\alpha+\delta}$ be k if $k \notin H$, and mk otherwise. Continuing in this fashion, the construction can be completed so that conditions (i), (ii) and (iii) hold.

Since κ is regular, there is some $\xi < \epsilon$ such that $g = (g_\alpha)_{\alpha<\lambda} \in M^{\mathbb{P}_\xi}$. We can now define a map $S : \lambda \to [\lambda]^{\omega_1}$ by $S(\alpha) = S_\alpha$ where S_α is defined as follows: For each $\varphi \in Aut(\Gamma)$ where $\Gamma \prec \Gamma_{\omega_1}$ is countable and dom φ is invariant under g_α, by (iii) we can choose $\beta_\varphi > \alpha$ so that we have $\beta_\varphi \notin \cup_{\delta<\alpha}S_\delta$ and $\varphi g_\alpha \varphi^{-1}(a) = g_{\beta_\varphi}(a)$ for all $a \in \text{ran } \varphi$. Let

$$S_\alpha = \{\beta_\varphi \mid \text{dom } \varphi \text{ is invariant under } g_\alpha\}.$$

Note that for $\alpha \neq \beta$, $S_\alpha \cap S_\beta = \emptyset$.
Without loss of generality, $\mathbb{Q}_{g,S}$ is the denotation of $\dot{\mathbb{Q}}_\xi$. Then $M^{\mathbb{P}_{\xi+1}}$ contains a filter $\mathcal{H}$ which is $\mathbb{Q}_{g,S}$-generic over $M^{\mathbb{P}_\xi}$.

Claim 1. For each $a \in \Gamma_{\omega_1}$, the set $D_a = \{\langle\varphi, F\rangle \in \mathbb{Q}_{g,S} \mid a \in \text{dom } \varphi\}$ is dense in $\mathbb{Q}_{g,S}$.

Proof. Let $\langle\varphi, F\rangle \in \mathbb{Q}_{g,S}$ be arbitrary. Suppose that we have dom $\varphi = \Gamma$ and dom $F = \{\alpha_n\}_{n<\omega}$. Extend φ to $h \in G$. Then $(g^h_{\alpha_n})_{n<\omega}$ is strongly generic, $g^h_{\alpha_n}[\varphi[\Gamma]] = g_{F(\alpha_n)}[\varphi[\Gamma]] = \varphi[\Gamma]$ for each $n < \omega$ and $(g^h_{\alpha_n} \restriction \varphi[\Gamma])_{n<\omega}$ is existentially closed. So, by Proposition 2.9 there is $k \in G_{\varphi[\Gamma]}$ such that $g^{hk}_{\alpha_n} = g_{F(\alpha_n)}$ for each $n < \omega$. Let $\psi = hk$. Then ψ extends φ, and for each $n < \omega$, $g^\psi_{\alpha_n} = g_{F(\alpha_n)}$. Using the Lowenheim-Skolem Theorem we can extend $\Gamma \cup \{a\}$ to a countable $\Gamma' \prec \Gamma_{\omega_1}$ such that Γ' is invariant under each g_{α_n} and ψ, and such that $(\Gamma', g_{\alpha_n} \restriction \Gamma')_{n<\omega}$ is an elementary substructure of $(\Gamma_{\omega_1}, g_{\alpha_n})_{n<\omega}$. Since the sequence $(g_{\alpha_n})_{n<\omega}$ is strongly generic, Proposition 2.8 shows that $(g_{\alpha_n} \restriction \Gamma')_{n<\omega}$ is existentially closed. Let $\tau = \psi \restriction \Gamma'$. Then $\langle\tau, F\rangle \leq \langle\varphi, F\rangle$. □

Claim 2. For each $\alpha < \lambda$, the set $K_\alpha = \{\langle \varphi, F\rangle \in \mathbb{Q}_{g,S} \mid \alpha \in \text{dom } F\}$ is dense in $\mathbb{Q}_{g,S}$.

Proof. Let $\langle \varphi, F\rangle \in \mathbb{P}$ be arbitrary. Suppose dom $F = \{\alpha_n\}_{n<\omega}$, and define Γ and ψ as before. Since the sequence $(g_{\alpha_n} \mid n < \omega)^\wedge g_\alpha$ is strongly generic we can use the Lowenheim Skolem Theorem and Proposition 2.8 to extend Γ to a countable random graph Γ' such that Γ' is invariant under each g_{α_n}, g_α and ψ, and such that $(g_{\alpha_n} \restriction \Gamma' \mid n < \omega)^\wedge g_\alpha \restriction \Gamma'$ is existentially closed. Let $\tau = \psi \restriction \Gamma'$. By the definition of S, we can find an appropriate $\beta > \alpha$ so that $\tau g_\alpha \tau^{-1} \restriction \Gamma' = g_\beta \restriction \Gamma'$. So $\langle \tau, F \cup (\alpha, \beta)\rangle \leq \langle \varphi, F\rangle$. □

$\mathcal{H}$ intersects each D_a and K_α. Let

$$\Psi = \bigcup\{\varphi \mid \langle \varphi, F\rangle \in \mathcal{H}\} \text{ and } \Phi = \bigcup\{F \mid \langle \varphi, F\rangle \in \mathcal{H}\}.$$

By Claim 2 and clauses (2) and (3) in Definition 5.3, $\Phi : \lambda \to \lambda$ is an injective function with $\Phi(\alpha) > \alpha$ for each $\alpha < \lambda$. By Claim 1, we have $\Psi \in Aut(\Gamma_{\omega_1}) = G$; and clearly $\Psi g_\alpha \Psi^{-1} = g_{\Phi(\alpha)}$ for each $\alpha < \lambda$. Since $\Psi \in G$, we have that $\Psi \in H_\alpha \subseteq H_{\xi_\alpha}$, for some $\alpha < \lambda$. But then we have $g_{\Phi(\alpha)} = \Psi g_\alpha \Psi^{-1} \in H_{\xi_\alpha}$, which is a contradiction. □

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