The Cofinality of the Random Graph

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Abstract

We show that under Martin's Axiom, the cofinality $cf(Aut(\Gamma))$ of the automorphism group of the random graph Γ is 2^ω.

1 Introduction

Suppose that G is a group which is not finitely generated. Then G can be expressed as the union of a chain of proper subgroups. The cofinality of G, written $cf(G)$, is defined to be the least cardinal λ such that G can be expressed as the union of a chain of λ proper subgroups. In [**6**], Macpherson and Neumann proved that $cf(Sym(\omega)) > \omega$. In [**7**], Sharp and Thomas proved that it is consistent that $cf(Sym(\omega))$ and 2^ω can be any two prescribed regular, uncountable cardinals, subject only to the obvious requirement that $cf(Sym(\omega)) \leq 2^\omega$. Then, in [**8**], Sharp and Thomas considered the relationship between $cf(Sym(\omega))$ and two well-known cardinal invariants of the continuum, the dominating number $\mathfrak{d}$ and the bounding number $\mathfrak{b}$. They proved that $cf(Sym(\omega)) \leq \mathfrak{d}$, and that both $cf(Sym(\omega)) < \mathfrak{b}$ and $\mathfrak{b} < cf(Sym(\omega))$ are consistent with ZFC.

If we regard $Sym(\omega)$ as the automorphism group of $\langle \omega; \rangle$, the "trivial countably infinite structure," then it is natural to try to compare $cf(Sym(\omega))$ and $cf(Aut(\mathcal{M}))$, where $\mathcal{M}$ is a countable structure. In [**9**], Thomas showed that if $\mathcal{M}$ is ω-categorical, then $cf(Aut(\mathcal{M})) \leq cf(Sym(\omega))$. There exist countable ω-categorical structures $\mathcal{M}$ such that $cf(Aut(\mathcal{M})) < cf(Sym(\omega))$.

For example, in [**5**], Lascar showed that there exists a countable ω-categorical structure $\mathcal{B}$ such that the product of countably many cyclic groups of order 2 is a homomorphic image of $Aut(\mathcal{B})$. It follows that $cf(Aut(\mathcal{B})) = \omega$. In [**9**], Thomas also showed that if $\mathcal{M}$ is a vector space over a finite field $\mathbb{F}$, then $cf(Aut(\mathcal{M})) = cf(Sym(\omega))$. On the other hand, the following question is open:

Question. Is it consistent that there exists a countable ω-categorical structure $\mathcal{M}$ such that

$$\omega < cf(Aut(\mathcal{M})) < cf(Sym(\omega))?$$

We denote by Γ the random graph (see [**1**, pp. 37-38]) which is uniquely characterized up to isomorphism among graphs on countably many vertices by the following property:

$(*)$ *If U, V are disjoint, finite sets of vertices in Γ, then there is a vertex x of Γ which is adjacent to all vertices in U and to no vertices in V.*

In [**2**], Hodges, Hodkinson, Lascar, and Shelah showed that $cf(Aut(\Gamma)) > \omega$. In this paper, it will be shown that under Martin's Axiom, $cf(Aut(\Gamma)) = 2^\omega$.

Definition 1.1.
(1) $MA(\lambda)$ is the statement that if $\mathbb{P}$ is a nonempty c.c.c partial order, and $\mathcal{D}$ a family of dense subsets of $\mathbb{P}$ with $|\mathcal{D}| \leq \lambda$, then there exists a filter $\mathcal{G} \subseteq \mathbb{P}$ such that $\mathcal{G} \cap D \neq \emptyset$ for every $D \in \mathcal{D}$.
(2) Martin's Axiom (MA) is the statement that $MA(\lambda)$ holds for all $\lambda < 2^\omega$.

The main result to be proved is the following:

Theorem 1.2. Let Γ be the random graph and let $G = Aut(\Gamma)$. If λ is a regular cardinal, then $MA(\lambda) \models cf(G) > \lambda$.

From now on $G = Aut(\Gamma)$. We will prove Theorem 1.2 in the following way. Suppose that $G = \cup_{\alpha<\lambda} H_\alpha$. We will use $MA(\lambda)$ to construct a "generic" sequence of automorphisms $(g_\alpha)_{\alpha<\lambda}$ (as defined in [**2**]and [**10**]) and a strictly increasing sequence $(\xi_\alpha)_{\alpha<\lambda}$ such that $g_\alpha \in H_{\xi_\alpha} \setminus \cup_{\beta<\xi_\alpha} H_\beta$ for each $\alpha < \lambda$.

We will then use $MA(\lambda)$ again to find an element $\Psi \in G$ such that for each $\alpha < \lambda$, there is a $\beta > \alpha$ such that $\Psi g_\alpha \Psi^{-1} = g_\beta$. Then for some $\alpha < \lambda$, $\Psi \in H_\alpha \subseteq H_{\xi_\alpha}$. So $g_\beta = \Psi g_\alpha \Psi^{-1} \in H_{\xi_\alpha}$, a contradiction. All of this will be made precise later in this paper.

Theorem 1.2 and the proof of Corollary 2.2 in [**7**] give us the following result:

Corollary 1.3. Let $M \models GCH$ and suppose that $\lambda \leq \theta$ are regular uncountable cardinals in M. Then there exists a c.c.c. poset $\mathbb{P}$ such that $M^{\mathbb{P}} \models cf(G) = \lambda \leq \theta = 2^\omega$.

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2 Generic Sequences of Automorphisms

In this section, following Truss [**10**], we define the notion of a generic sequence of automorphisms. The existence of generic sequences of countable length was proved in [**2**]. Since many of the ideas in [**2**] are central to the proof of Theorem 1.2, we repeat the relevant results in this section. However, we will rewrite this exposition in the language of games, as we feel that it is slightly easier to understand in this form. We then show that under $MA(\lambda)$, we can construct generic sequences of length λ.

If $A \subseteq \Gamma$, then we denote by G_A the pointwise stabilizer of A. It is well known that G is a Polish group with basis

$$\{gG_F \mid g \in G,\ F \subseteq \Gamma,\ |F| < \omega\}.$$

The open subgroups of G are precisely the subgroups which contain the pointwise stabilizer of a finite set.

Lemma 2.1. If H is an open subgroup of G, then there are only finitely many subgroups K of G that contain H.

Proof. See [**2**, Lemma 2.4].

In particular, if $G = \cup_{\alpha<\lambda} H_\alpha$, then H_α is not open for each $\alpha < \lambda$.

Definition 2.2.
(1) If $g, h \in G$, then the conjugate of g by h is defined to be $g^h = hgh^{-1}$.
(2) If $(g_1, \ldots, g_n) \in G^n$, then the corresponding conjugacy class is defined to be

$$(g_1, \ldots, g_n)^G = \{(g_1^h, \ldots, g_n^h) \mid h \in G\}.$$

Recall that a subset $C \subseteq G$ is *comeagre* if C contains a countable intersection of dense open subsets of G. A subset $M \subseteq G$ is *meagre* if $G \setminus M$ is comeagre.

Definition 2.3. $(g_1, \ldots, g_n) \in G^n$ is said to be *generic* if $(g_1, \ldots, g_n)^G$ is comeagre in G^n (in the product topology).

Note that there can be at most one generic conjugacy class, for if $(g_1, \ldots, g_n)^G$ and $(h_1, \ldots, h_n)^G$ are both comeagre, then they must intersect. But conjugacy classes are either disjoint or coincide. So $(g_1, \ldots, g_n)^G = (h_1, \ldots, h_n)^G$.

Notation. $(\Gamma, g_1, \ldots, g_n)$ is the obvious expansion of Γ to a language with n new 1-place function symbols.

Proposition 2.4. Let $(g_1, \ldots, g_n)$, $(h_1, \ldots, h_n) \in G^n$. Then

$$(h_1, \ldots, h_n) \in (g_1, \ldots, g_n)^G \text{ iff } (\Gamma, g_1, \ldots, g_n) \simeq (\Gamma, h_1, \ldots, h_n).$$

In particular, if $(g_1, \ldots, g_n)$, $(h_1, \ldots, h_n)$ are both generic sequences, then $(\Gamma, g_1, \ldots, g_n) \simeq (\Gamma, h_1, \ldots, h_n)$.

Proof. $(\Gamma, g_1, \ldots, g_n) \simeq (\Gamma, h_1, \ldots, h_n)$ iff there is a $k \in G$ such that $k(g_i(a)) = h_i(k(a))$ for all $a \in \Gamma$ and $i = 1, \ldots, n$ iff there exists $k \in G$ such that $h_i = g_i^k$ for all $i = 1, \ldots, n$. □

We will use Banach-Mazur games to show that there is a generic conjugacy class. For each $A \subseteq G$ let $\mathfrak{G}(A)$ be the game defined as follows. Let

$$\mathbb{P} = \{f : \Gamma \to \Gamma \mid f \text{ is a finite partial isomorphism}\}$$

where $\mathbb{P}$ is ordered by $p \leq q$ iff $p \supseteq q$. Then in the game $\mathfrak{G}(A)$, Players I and II choose a decreasing sequence

$$p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots, n \in \omega$$

of elements of $\mathbb{P}$. Player I chooses p_i iff i is even. Player II wins iff $\cup_{n\in\omega} p_n \in A$.

Remark. Player II can easily ensure that $\cup_{n\in\omega} p_n \in G$.

Theorem 2.5. Player II has a winning strategy in $\mathfrak{G}(A)$ iff A is comeagre in G.

Proof. See [**4**, Theorem 8.33].

Theorem 2.6. Player I has a winning strategy in $\mathfrak{G}(A)$ iff there is a finite $F \subseteq \Gamma$ and $g \in G$ such that $A \cap gG_F$ is meagre in gG_F.

Proof. See [**4**, Theorem 8.33].

Corollary 2.7. $g \in G$ is generic iff Player II has a winning strategy in $\mathfrak{G}(g^G)$.

By the next result, if H is not open, then Player I does not have a winning strategy in $\mathfrak{G}(G \setminus H)$. This will allow us to construct a generic automorphism which is not in H.

Theorem 2.8. Let H be a subgroup of G. Then H is open iff Player I has a winning strategy in $\mathfrak{G}(G \setminus H)$.

Proof. If H is open, then there is a finite $F \subseteq \Gamma$ such that $G_F \leq H$. As his first move, let Player 1 play $p_0 = id \restriction F$. Then clearly Player I can ensure that $\cup_{n\in\omega} p_n \in G_F$.

Conversely, suppose that Player I has a winning strategy in $\mathfrak{G}(G \setminus H)$. By Theorem 2.6, there is a finite $F \subseteq \Gamma$ and $g \in G$ such that $(G \setminus H) \cap gG_F$ is meagre in gG_F. Then $H \cap gG_F$ is comeagre in gG_F. In particular, we have that $H \cap gG_F \neq \emptyset$. So, we may assume that $g \in H$. Therefore, $H \cap gG_F = g(H \cap G_F)$. It follows that $H \cap G_F$ is comeagre in G_F, and hence the same is true of each coset of $H \cap G_F$ in G_F. Since distinct cosets are disjoint, we have that $G_F = H \cap G_F$. Thus $G_F \leq H$, and so H is open. □

We now show that a generic conjugacy class exists. Consider the class $\mathcal{A}$ of all structures of the form $\langle X, f\rangle$ where X is a finite graph, and $f \in Aut(X)$. We say that $\langle Y, g\rangle$ is a substructure of $\langle X, f\rangle$ iff $Y \subseteq X$ and $g = f \restriction Y$.

Clearly $\mathcal{A}$ has the amalgamation property, and hence there exists a unique countable universal homogeneous structure $\langle \Delta, \varphi \rangle$ with respect to $\mathcal{A}$.

Claim 2.9. Δ is the random graph.

To prove this claim, we will need the following theorem of Hrushovski.

Theorem 2.10.(Hrushovski) Let X be a finite graph. Then there exists a finite graph Z, containing X as an induced subgraph, such that any isomorphism between induced subgraphs of X extends to an automorphism of Z.

Proof. See [**3**].

Proof of Claim 2.9. Suppose that $\langle X, f \rangle \in \mathcal{A}$ and that $X = U \cup V$ is a partition. Let $Y = X \cup \{x\}$, where x is a new vertex which is adjacent to all the vertices in U and to none in V. By Hrushovski's Theorem, there exists a finite graph $Z \supseteq Y$ such that any isomorphism between subgraphs of Y extends to an automorphism of Z. In particular, f extends to an automorphism g of Z and $\langle Z, g \rangle \in \mathcal{A}$. Claim 2.9 follows easily. □

Theorem 2.11. φ is a generic element of G.

Proof. Consider a play of the game $\mathfrak{G}(\varphi^G)$, say

$$p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots, n \in \omega,$$

and let $f = \cup_{n \in \omega} p_n$. It is clear that Player II can play so that the following hold:
(1) $f \in G$.
(2) for all odd i, dom $p_i =$ ran p_i. (Use Hrushovski's Theorem).
(3) (Γ, f) is existentially closed in the class of locally-$\mathcal{A}$ structures.
It follows that (Γ, f) is a universal homogeneous structure with respect to $\mathcal{A}$. Thus $(\Gamma, f) \simeq (\Gamma, \varphi)$. So by Proposition 2.4, $f \in \varphi^G$. □

We can easily generalize the above to generic sequences $(g_1, \ldots, g_n) \in G^n$ for each $1 \leq n < \omega$. Let $\mathbb{P}^n = \mathbb{P} \times \cdots \times \mathbb{P}$. Then for each subset $A \subseteq G^n$, we have the obvious game $\mathfrak{G}(A)$.

Theorem 2.12. Player II has a winning strategy in $\mathfrak{G}(A)$ iff A is comeagre in G^n (in the product topology).

Theorem 2.13. Player I has a winning strategy in $\mathfrak{G}(A)$ iff there are finite $F_1, \ldots, F_n \subseteq \Gamma$ and $g_1, \ldots, g_n \in G$ such that $A \cap (g_1 G_{F_1} \times \cdots \times g_n G_{F_n})$ is meagre in $g_1 G_{F_1} \times \cdots \times g_n G_{F_n}$.

Corollary 2.14. $(g_1, \ldots, g_n) \in G^n$ is generic iff Player II has a winning strategy in $\mathfrak{G}((g_1, \ldots, g_n)^G)$.

Consider the class $\mathcal{A}_n$ of all structures of the form $\langle X, f_1, \ldots, f_n \rangle$ where X is a finite graph, and $f_i \in Aut(X)$ for each $i = 1, \ldots, n$. As before, we have the obvious notion of a substructure, and since $\mathcal{A}_n$ has the amalgamation property, there exists a unique countable universal homogeneous structure $\langle \Delta, \varphi_1, \ldots, \varphi_n \rangle$ with respect to $\mathcal{A}_n$. Using the same arguments as before, we see that Δ is the random graph, and that $(\varphi_1, \ldots, \varphi_n)$ is generic. We therefore have the following:

Theorem 2.15. There exist generic $(g_1, \ldots, g_n) \in G^n$ for each $1 \leq n < \omega$.

Theorem 2.16. If $(g_1, \ldots, g_n)$ is generic, then

$$A = \{g \in G \mid (g_1, \ldots, g_n, g) \text{ is generic}\}$$

is comeagre in G.

Proof. See [**4**, Theorem 8.33].

To prove Theorem 1.2, we will need longer generic sequences.

Definition 2.17. If β is any ordinal, then we say that $(g_\alpha)_{\alpha<\beta}$ is generic if for any $n \in \omega$ and $\alpha_1 < \cdots < \alpha_n < \beta$, $(g_{\alpha_1}, \ldots, g_{\alpha_n}) \in G^n$ is generic.

Theorem 2.18. **($MA(\lambda)$).** Let $\beta < \lambda$ and $(g_\alpha)_{\alpha<\beta}$ be generic. Then, $\{g_\beta \in G \mid (g_\alpha)_{\alpha \leq \beta} \text{ is generic}\}$ is comeagre in G.

Proof. $\{g_\beta \in G \mid (g_\alpha)_{\alpha \leq \beta} \text{ is generic}\}$ is the intersection of the sets

$$\{f \in G \mid (g_{\alpha_1}, \ldots, g_{\alpha_n}, f) \text{ is generic}\} \ (n \in \omega,\ \alpha_1 < \ldots < \alpha_n < \beta)$$

This is an intersection of $< \lambda$ comeagre sets and hence is comeagre by $MA(\lambda)$. □

The next lemma will enable us to construct our desired generic sequence as stated in the Introduction.

Lemma 2.19. Let C be comeagre in G, H a subgroup of G which is not open, and $\varphi : \Gamma \to \Gamma$ a finite partial isomorphism. Then there is a $g \in C \setminus H$ which extends φ.

Proof. Consider a play of the game $\mathfrak{G}(C \setminus H)$, say

$$p_0 \geq p_1 \geq \ldots \geq p_n \geq \ldots, n \in \omega.$$

Let Player I use the following "strategy": On his first move Player I plays $p_0 = \varphi$. For the rest of the game, Player I pretends to be Player II using a winning strategy in the game $\mathfrak{G}(C)$; and thus Player I will ensure that $g = \cup_{n<\omega} p_n \in C$. By Theorem 2.8, this strategy is not winning for Player I in $\mathfrak{G}(G \setminus H)$. Hence Player II can ensure that $g = \cup_{n<\omega} p_n \in G \setminus H$. □

3 Proof of Theorem 1.2

We are now ready to prove the main result of this paper.

Theorem 1.3. Let Γ be the random graph and let $G = Aut(\Gamma)$. If λ is a regular cardinal, then $MA(\lambda) \models cf(G) > \lambda$.

Proof. Let $\kappa \leq \lambda$ be a regular cardinal, and suppose that $G = \cup_{\alpha<\kappa} H_\alpha$ is a chain of proper subgroups. Then by **[2]**, $\kappa > \omega$. By Lemma 2.1, each H_α is not open. We define a sequence $(g_\alpha)_{\alpha<\kappa}$ of elements of G and a strictly increasing sequence $(\xi_\alpha)_{\alpha<\kappa}$ of ordinals so that:

(i) For each $\alpha < \kappa$, $g_\alpha \in H_{\xi_\alpha} \setminus \cup_{\beta<\xi_\alpha} H_\beta$.

(ii) For each $\alpha < \kappa$, $(g_\beta)_{\beta<\alpha}$ is generic.

(iii) Suppose that $\gamma \leq \alpha < \kappa$ and that $\varphi : \Gamma \to \Gamma$ is a finite partial isomorphism such that dom φ is invariant under g_γ. Then there exists $\beta > \alpha$ such that $\varphi g_\gamma \varphi^{-1}(a) = g_\beta(a)$ for all $a \in$ ran φ.

As the set of generic elements of G is comeagre in G, we can choose a generic element $g_0 \in G$. Let ξ_0 be the least ordinal such that $g_0 \in H_{\xi_0}$. Then g_0, ξ_0 satisfy (i) and (ii).

Now suppose that $(g_\beta)_{\beta<\alpha}$ has been defined satisfying (i) and (ii). Let J be the set of all pairs $\langle \gamma, \varphi \rangle$ satisfying the following conditions:

(a) $\gamma < \alpha$.
(b) $\varphi : \Gamma \to \Gamma$ is a finite partial isomorphism.
(c) dom φ is invariant under g_γ.

Then J has cardinality $\mu < \kappa$. Enumerate this set as $(\langle \gamma_\beta, \varphi_\beta \rangle)_{\alpha \leq \beta < \alpha+\mu}$, and for each β, let $h_\beta = \varphi_\beta g_{\gamma_\beta} \varphi_\beta^{-1} \restriction \text{ran } \varphi_\beta$. We extend each $h_{\alpha+\delta}$ to $g_{\alpha+\delta}$ satisfying (i) and (ii) as follows:

Suppose $(g_\beta)_{\beta<\alpha+\delta}$ and $(\xi_\beta)_{\beta<\alpha+\delta}$ satisfy (i) and (ii). The set

$$C = \{g_{\alpha+\delta} \in G \mid (g_\beta)_{\beta \leq \alpha+\delta} \text{ is generic}\}$$

is comeagre in G. Let $\rho = sup\{\xi_\beta \mid \beta < \alpha + \delta\}$. By Lemma 2.19 there is $g_{\alpha+\delta} \in C \setminus H_\rho$ such that $g_{\alpha+\delta}$ extends $h_{\alpha+\delta}$. Let $\xi_{\alpha+\delta}$ be the least ordinal such that $g_{\alpha+\delta} \in H_{\xi_{\alpha+\delta}}$. Continuing in this fashion, the construction can be completed so that conditions (i), (ii) and (iii) hold.

Now define a map $S : \kappa \to [\kappa]^\omega$ by $S(\alpha) = S_\alpha$ where S_α is defined as follows: For each finite partial isomorphism $\varphi : \Gamma \to \Gamma$ with dom φ invariant under g_α, by (iii) we can choose $\beta_\varphi > \alpha$ so that we have $\beta_\varphi \notin \cup_{\delta<\alpha} S_\delta$ and $\varphi g_\alpha \varphi^{-1}(a) = g_{\beta_\varphi}(a)$ for all $a \in$ ran φ. Let

$$S_\alpha = \{\beta_\varphi \mid \text{dom } \varphi \text{ is invariant under } g_\alpha\}.$$

Note that for $\alpha \neq \beta$, $S_\alpha \cap S_\beta = \emptyset$.

Let $\mathbb{P}$ be the partial order consisting of all conditions $\langle \varphi, F \rangle$ such that:
(1) $\varphi : \Gamma \to \Gamma$ is a finite partial isomorphism
(2) $F : \kappa \to \kappa$ is a finite injective partial map
(3) For all $\alpha \in$ dom F, $F(\alpha) \in S_\alpha$
(4) For all $\alpha \in$ dom F, dom φ is invariant under g_α.
(5) For all $\alpha \in$ dom F and $a \in$ ran φ, $\varphi g_\alpha \varphi^{-1}(a) = g_{F(\alpha)}(a)$
$\mathbb{P}$ is ordered by $\langle \varphi_1, F_1 \rangle \leq \langle \varphi_2, F_2 \rangle$ iff $\varphi_1 \supseteq \varphi_2$ and $F_1 \supseteq F_2$.

Claim 1. $\mathbb{P}$ is a c.c.c partial order.

Proof. To prove that $\mathbb{P}$ is c.c.c, suppose that $\{\langle \varphi_\xi, F_\xi \rangle \mid \xi < \omega_1\}$ is an antichain. Without loss of generality, $\varphi_\xi = \varphi$ for all ξ. Let $A_\xi =$ dom F_ξ. By the Δ-system lemma, we may assume that $\{A_\xi \mid \xi < \omega_1\}$ forms a Δ-system with root R. For each $\alpha \in R$, we must have $F_\xi(\alpha) \in S_\alpha$. Thus there are only countably many possibilities for $F_\xi \restriction R$. Therefore there is an uncountable $X \subseteq \omega_1$ such that $F_\xi \restriction R$ are all the same for all $\xi \in X$. It follows that $\{\langle \varphi, F_\xi \rangle \mid \xi \in X\}$ are pairwise compatible, a contradiction. □

Claim 2. For each $a \in \Gamma$, the set $D_a = \{\langle \varphi, F \rangle \in \mathbb{P} \mid a \in \text{dom } \varphi\}$ is dense.

Proof. Let $\langle \varphi, F \rangle \in \mathbb{P}$ be arbitrary. Suppose dom $F = \{\alpha_1, \ldots, \alpha_n\}$. Extend φ to $h \in G$. Then $(g^h_{\alpha_1}, \ldots, g^h_{\alpha_n})$ is generic and agrees with $(g_{F(\alpha_1)}, \ldots, g_{F(\alpha_n)})$ on ran φ. By Proposition 2.4 $(\Gamma, g^h_{\alpha_1}, \ldots, g^h_{\alpha_n}) \simeq (\Gamma, g_{F(\alpha_1)}, \ldots, g_{F(\alpha_n)})$. Since this structure is homogeneous with respect to $\mathcal{A}_n$ and the generic sequences agree on ran φ, we can find an automorphism $k \in G_{\text{ran } \varphi}$ such that for each $i = 1, \ldots, n$, $g^{hk}_{\alpha_i} = g_{F(\alpha_i)}$. Let $\psi = hk$. Then ψ extends φ, and for each $i = 1, \ldots, n$, $g^{\psi}_{\alpha_i} = g_{F(\alpha_i)}$. Since the sequence $(g_{\alpha_1}, \ldots, g_{\alpha_n})$ is generic we can extend dom $\varphi \cup \{a\}$ to a finite set Y such that Y is invariant under $g_{\alpha_1}, \ldots, g_{\alpha_n}$. Let $\tau = \psi \restriction Y$. Then $\langle \tau, F \rangle \leq \langle \varphi, F \rangle$. □

Claim 3. For each $a \in \Gamma$, the set $E_a = \{\langle \varphi, F \rangle \in \mathbb{P} \mid a \in \text{ran } \varphi\}$ is dense.

Proof. Define ψ as before. Extend dom $\varphi \cup \{\psi^{-1}(a)\}$ to a finite set Y such that Y is invariant under $g_{\alpha_1}, \ldots, g_{\alpha_n}$. Let $\tau = \psi \restriction Y$. Then $\langle \tau, F \rangle \leq \langle \varphi, F \rangle$. □

Claim 4. For each $\alpha < \kappa$, the set $K_\alpha = \{\langle \varphi, F \rangle \in \mathbb{P} \mid \alpha \in \text{dom } F\}$ is dense.

Proof. Let $\langle \varphi, F \rangle \in \mathbb{P}$ be arbitrary. Suppose dom $F = \{\alpha_1, \ldots, \alpha_n\}$, and define ψ as before. Since the sequence $(g_{\alpha_1}, \ldots g_{\alpha_n}, g_\alpha)$ is generic we can extend dom φ to a finite set Y such that Y is invariant under $g_{\alpha_1}, \ldots, g_{\alpha_n}, g_\alpha$. Let $\tau = \psi \restriction Y$. By the definition of S, we can find an appropriate $\beta > \alpha$ so that $\tau g_\alpha \tau^{-1} \restriction \text{ran } \tau = g_\beta \restriction \text{ran } \tau$. So $\langle \tau, F \cup (\alpha, \beta) \rangle \leq \langle \varphi, F \rangle$. □

Now let $\mathcal{G}$ be a filter intersecting each D_a, E_a and K_α. Let

$$\Psi = \bigcup \{\varphi \mid \langle \varphi, F \rangle \in \mathcal{G}\} \text{ and } \Phi = \bigcup \{F \mid \langle \varphi, F \rangle \in \mathcal{G}\}.$$

By Claim 4, $\Phi : \kappa \to \kappa$ is a strictly increasing function. By claims 2 and 3, $\Psi \in Aut(\Gamma) = G$; and clearly $\Psi g_\alpha \Psi^{-1} = g_{\Phi(\alpha)}$ for each $\alpha < \kappa$. Since

$\Psi \in G$, we have that $\Psi \in H_\alpha \subseteq H_{\xi_\alpha}$, for some $\alpha < \kappa$. But then we have $g_{\Phi(\alpha)} = \Psi g_\alpha \Psi^{-1} \in H_{\xi_\alpha}$, which is a contradiction. □

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