PHYSICS MASTERY
for the Advanced High School Student

Complete Physics Review
with 400 SAT and AP Physics Questions

by Dr. Tony Rothman
Edited by Dr. Steve Warner
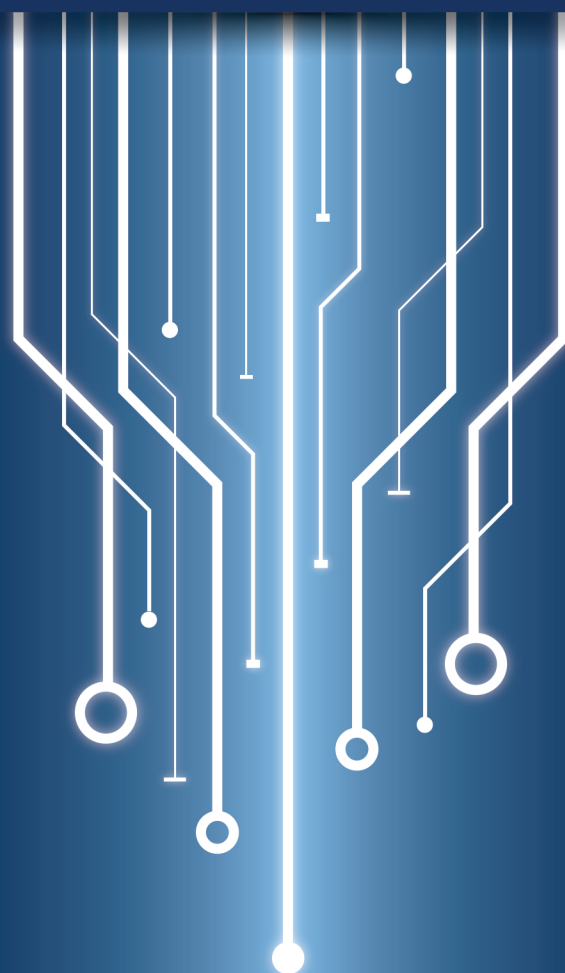
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GENERAL STRATEGIES FOR
SOLVING PHYSICS PROBLEMS

*P*hysics Mastery has been designed to help you ace the SAT physics subject test and the AP physics 1 and AP physics 2 tests. To that end, the book is divided into review sections and problem sections. The review sections are, as their name implies, reviews. They are intended mostly to refresh your memory about material you should already know—at least have already seen—as well as to familiarize you with our notation and style. They do not replace a good text book (though they may replace a bad one) and the topics chosen are limited to those you will encounter on the College Board exams; they cannot and do not cover as much ground as a serious first-year physics course.

The last statement is a slight lie. In addition to reviewing the material needed to perform well on the subject tests and AP physics exams, we also present some material that is a bit more difficult than that which you will find on standardized tests. The hope is that mastering the more difficult material will make those tests easier for you.

For most people physics is a difficult subject, and nothing can replace actually understanding the material, which requires prolonged study and practice. Rote memorization at the eleventh hour is doomed to failure; memorization neither tells you which concepts are important and what equations to use, nor how to recognize a mistake. We hope that the problem-solving approach you will encounter here—which sometimes differs considerably from the methods taught in high school—will stay with you for the remainder of your careers. If you can do all the problems in *Physics Mastery*, you should perform very well on the College Board exams.

Broadly, we have attempted to keep the tone of the book conversational but concise. The SAT physics subject test and the AP physics 1 and AP physics 2 exams do not require calculus. Therefore, we do not either, although we have tried to be as rigorous as possible within that limitation, and have occasionally resorted to footnotes to explain concepts that really do require calculus for proper understanding.

The overall map of *Physics Mastery* is simple: easy to hard. In each chapter the review sections proceed from easier material to more difficult material. The chapters themselves proceed from elementary subjects to more demanding topics and the problems also proceed from basic subject test level questions to challenging AP-level problems. Admittedly, this scheme has been difficult to carry out in a consistent fashion because the division between subject test material and AP-level on the exams is not entirely clear and the same topics frequently appear on both exams. As a rule, however, subject test questions are more qualitative (conceptual), while the AP-level questions expect quantitative (mathematical) responses. The subject-level student is advised to read as much of the review material as possible, in particular the introductions to each section, which are mostly qualitative. To make life easier, each chapter is headed with a "warning label" as to which sections are at the subject test level and which are at the AP level. If you believe you have already mastered the material, you may proceed directly to the problems and consult the review sections as necessary.

The problems have been chosen to closely resemble those found on the exams, but we do not guarantee that you will find these exact problems on any exam. Most are standard exercises that you will encounter, not only on the College Board tests, but in any physics course. However, we give innumerable examples as well as solutions to every problem. Furthermore, for many problems we give multiple solutions: a quick solution, which may amount to an educated guess and which often involves **POE** (process of elimination), followed by a rigorous solution, sometimes an alternative solution, and occasionally helpful remarks that will undoubtedly prove to be just that. We have also found many of the College Board problems unclear or downright incorrect and have not shied away from fixing them. Throughout the review sections we also give hints, tips and advice, either boxed or in **boldface** that should make solving any problem easier. The most important strategies for problem solving we reveal right now:

Strategy 1: Draw a Diagram

As a rule, the first thing one should *always* do in solving a physics or math problem is make a good diagram. The SAT physics subject test usually provides these already, and so you are saved the trouble. If there is no diagram, draw one, even if it is just a vector pointing in the correct direction: ⟶

Strategy 2: Eliminate and Guess

SAT physics subject test questions are multiple choice and it pays to make a guess if you can eliminate most of the answers. If a problem has you going this way for 5 kilometers, that way for another 4, and a third way for 7, the final distance from your starting point cannot be 100 kilometers. If that is a choice, cross it off. Various ways you can eliminate answers are suggested in the other strategies below and in the problems.

Strategy 3: Avoid Excessive Calculator Use

Contrary to popular belief, the calculator is not a good way to eliminate answers. SAT subject test questions are conceptual and, at most, require only a few simple arithmetic operations. Calculators are superfluous (not to mention not allowed). Put yours away.

Because the questions are conceptual, logic is a useful weapon to eliminate incorrect choices. It may be that two choices are equally true or equally false, in which case neither can be the single correct response. Here is an example involving a box sliding down a frictionless ramp and hitting a spring (see problem 3.69).

Example 0.1: The maximum velocity of the box occurs at the instant when

(A) the box hits the spring.
(B) the potential energy is minimum.
(C) the kinetic energy is maximum.
(D) the forces on the block are zero.
(E) all of the above except (A)

Even without knowing exactly what the problem is about, you can arrive at the correct answer. By definition, maximum velocity implies that kinetic energy is a maximum. But because energy is conserved, (B) and (C) mean exactly the same thing. Therefore, there are at least two correct responses to the question and, logically, the only possible answer to the question is (E).

Strategy 4: Dimensional Analysis

In terms of physics itself, the most powerful weapon for checking answers (and one that you should use routinely in *every* step of *every* calculation you do from now until the day you die) is what physicists term **dimensional analysis**. Section 1.2, and a few of the problems in Chapter 1 give more detail on this crucial technique, but here is the basic idea:

Any quantity in physics comes with a set of dimensions—how it is expressed in terms of the fundamental physical quantities of mass, length and time.

Example 0.2: A spring with spring constant k is attached to a mass m that is oscillating in Earth's gravitational field, which produces an acceleration g. Which of the following expressions might be the spring's period?

(A) k/m
(B) k/gm
(C) $\sqrt{m/k}$
(D) $\sqrt{g^2/km}$
(E) (B) or (D)

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You might remember the answer, but let's say you don't. You can decide from the basics. First, you tell yourself that the period of anything is a time, t, so whatever your answer is, it had better have dimensions of t, or it doesn't have a chance of being correct. You might remember Hooke's law (Section 3.5), which says that the force exerted by a spring is $F = -kx$. You certainly remember $F = ma$, and that an acceleration has the dimensions of length/(time^2) $= \ell/t^2$. Thus, force has the dimensions of $m\ell/t^2$. Since x is also a length ℓ, dimensionally we have $k\ell = m\ell/t^2$. Hence, k/m has dimensions of $1/t^2$ and therefore m/k has dimensions of t^2. We want t, and so the correct answer must be (C).

You might wonder whether any of the other answers can be correct. **Hint: Whenever a question asks, "given two (or three) quantities, make a quantity with dimensions of such-and-such," there is always a single, unique answer.** (C) is the only possibility. The hint also means it is impossible for both (B) and (D) to be correct at the same time, ruling out (E) immediately.

Strategy 5: Think About the Numbers

A second powerful tool in eliminating incorrect physical choices is a number sense. It is useful to create a **number bank** with which you can perform a **reality check** on your answers.

Example 0.3: A teacher is recording her mass with a precision of one-half percent. What is the most likely result?

        (A) 6.43 kg
        (B) 60 kg
        (C) 64.3 kg
        (D) 600 kg
        (E) 643 kg

Now, only two of these candidates lie within a plausible mass range for an adult human. If you have a sense of what a realistic mass for a person is, you can eliminate all but choices (B) and (C) at once. Similarly, it is useful to have at your fingertips a sense about important sizes (continents, the Earth, the distance to the Moon, the diameter of the Sun, the size of an atom, and so on) as well as a sense of masses of the various particles that appear in physics (atoms, electrons, protons, people, etc.).

Strategy 6: Estimate Answers

Having a good number sense goes hand in hand with being able to make an estimate.

Example 0.4: Calculate the Earth's mean density given that the Earth's gravitational acceleration is $g = 9.8 \text{ m/s}^2$, its equatorial radius is $R = 6.378 \times 10^6 \text{ m}$ and the gravitational constant is $G = 6.67 \times 10^{-11} \ m^3/(\text{kg s}^2)$.

You might say that the Earth is a sphere with mass $M = \rho V$, density ρ and volume V. Then

$$g = \frac{GM}{R^2} = \frac{G\rho V}{R^2} = \frac{G\rho \left(\frac{4}{3}\pi R^3\right)}{R^2} = G\rho \left(\frac{4}{3}\pi\right) R$$

At that point you could plug all your numbers into your calculator and find that

$$\rho \approx 5500 \text{ kg/m}^3 = 5.5 \text{ gm/cm}^3.$$

On the other hand, $g \approx 10 \text{ m/s}^2$, where $\approx$ means "approximately equal to." Of course $\pi \approx 3$. Then

$$\rho \approx \frac{g}{4GR} \approx \frac{10}{4 \times 40 \times 10^{-5}} = \frac{10^6}{160} = 6250 \text{ kg/m}^3 \approx 6 \text{ gm/cm}^3.$$

For all the effort in putting in all those decimal points, the calculator's answer is hardly better than a quick estimate. *This density is a good number to put in your number bank*: Most of the things we encounter in daily life—water, wood, dirt, rocks, metal—have a density in the range $1 - 10\,\text{gm/cm}^3$, and if you got an answer that was ten times larger or smaller, a reality check would tell you that something went wrong.

AP problems involve more elaborate calculations than SAT subject test problems, but you should nevertheless be able to use your number sense to approximate the quantities involved, round off numbers and arrive at an answer that is within a factor of ten or better than one that you'd get with your calculator. We call this making an order-of-magnitude estimate (traditionally a "back-of-the-envelope calculation"). If your estimate and your numerical answer do not agree, you should rework the problem until they do. When you are finished, you should always subject your answer to a reality check; if the answer does not accord with your sense of reality, redo the problem. On an exam you probably do not have time to make careful estimates, but always perform a brief mental reality check to make sure the answer is plausible.

Strategy 7: Use Proportions

Often SAT and AP problems ask, "by what factor does such-and-such a quantity change if you do so-and-so." That wording is exam code to use proportions. The following problem actually appeared on an exam at Princeton:

Example 0.5: Transmitted electrical power is $P_t = IV$, where I is the current and V is the voltage. Most electricity to our homes is transmitted at about 100,000 volts. On the other hand, wires dissipate electrical power as heat according to $P_d = I^2R$, where R is the resistance of the wire. Resistance in turn is given by $R = \rho L/A$, where ρ is a property of the material called resistivity, L is the length of the wire and A is the wire's cross-sectional area.

The resistance per kilometer of the wire is, say, 1 ohm and the radius of the wire is 1 cm. If the voltage is lowered from 100,000 to 100 volts and you want to keep the transmitted power P_t and P_d the same as they were previously, how much does the weight of the wire change?

The vast majority of the students used their calculators to compute the area, compute the resistance, find a new current and then recalculate everything and get a numerical answer that was wrong. All that is required is proportions: If the voltage drops from 100,000 to 100 V, a factor of 1000, to keep P_t constant requires that I go up by the same factor of 1000. $P_d = I^2R$, and so to keep P_d constant you must then lower R by a factor of 1 million. For constant ρ, the resistance per kilometer is proportional to $1/A$; thus, you must increase the area by a factor of 1 million. The weight per kilometer of the wire is proportional to the area, and so the weight also goes up by a factor of one million. This is why we transmit electrical power at high voltage.

Never work out a problem twice if you can avoid it. Form a ratio of the required quantities and check the factor by which the answer changes.

All of these strategies point to the **Prime Directive** in physics:

> **Never Do a Calculation Until You Already Know the Answer**

It takes a lot of practice to be able to see an answer before you begin calculations, but to give you that practice is the purpose of this book.

1. PHYSICS BASICS REVIEW

Sections 1-4 of this chapter are at the subject-test level. Sections 5-7 are at the AP level. However, use of the right-hand-rule from Section 7 is required for the subject test.

1. Essential Terms

The word **physics** is from the Greek, *physika* meaning "natural things," and physics is the study of the material universe. Physics is the most basic science. That is why you love it.

All quantities in physics come with a set of **dimensions**, which tell you how it is expressed in terms of the fundamental quantities of **mass**, **length**, and **time**. Every dimension can be expressed in many different **units**. For example, mass can be measured in grams or kilograms. Most introductory texts, including this one, use the **Système Internationale**, otherwise known as the **MKS** (meter-kilogram-second) system, although most physicists avoid it.

All quantities in elementary physics can also be classified into two groups: **scalars** and **vectors**. Scalars are entirely characterized by their size, or magnitude (speed, temperature), while vectors are characterized by their **magnitude** and **direction** (velocity, acceleration).

Few SAT and AP problems ask direct questions about dimensions, units, scalars and vectors; however, the exams do expect you to recognize the meaning of the units when you encounter them. They also underlie the remaining material in this book, so we review them.

2. Dimensions and Units

All quantities in an elementary physics course and on an SAT or AP exam can be derived from five fundamental quantities: mass (M), length (L), time (T), charge (Q) and temperature (Θ). The last two appear only in problems involving electricity and thermodynamics. These five fundamental quantities are termed **dimensions**, in analogy to length, width and height. For example, since velocity is measured as a $\text{distance}/\text{time}$, we say the dimensions of velocity are length divided by time. Often this is written $[v] = \text{L/T}$ or $[v] = \text{LT}^{-1}$, the square brackets denoting dimensions.[1] In terms of problem solving, absorb the

> ### Supreme Hint:
>
> **The dimensions of your answer must be correct. If the dimensions are wrong, the answer cannot be right.**

If the numerical value of your answer is incorrect, this is a mistake. If the dimensions are wrong, the answer is meaningless. It is not even wrong. *You can often rule out answers on standardized tests by checking dimensions.* Some answers may have incorrect dimensions.

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[1] Although the use of upper-case letters for dimensions has become the "official" method, it can be confusing, not least because T is also used for temperature and writing m for a mass but M for dimensions of mass in the same line does not elucidate much. Most physicists in fact use lower case m, ℓ and t for the dimensions of mass, length and time, as we sometimes will, unofficially. See problem 7 below for a comparison of both methods. We also speak of the "dimensions of force" or "dimensions of energy," meaning quantities that have the dimensions of force or energy, even though these are not one of the five fundamental dimensions.

Although the dimensions of velocity are always length/time, there are any number of different **units** in which the dimensions can be measured. For instance, length can be measured in feet, miles, kilometers, centimeters, meters, furlongs… Time can be measured in hours, days, years, seconds, blinks of an eye…

> ### Semi-Supreme Hint:
>
> ### The units of your answer must be consistent. If they are not,
> ### you cannot get the correct numerical answer.

Although most physicists hate it, introductory texts and SAT exams use the metric "Système Internationale," or SI, units. When only distance, mass and time appear, this is the "MKS," or meter-kilogram-second, system.

Dimension	Unit
length	meter (m)
mass	kilogram (kg)
time	second (s)

All basic physics involves forming quantities from these fundamental ones, determining whether they are important and learning how they behave.

Dimensions	Units	Quantity
L/T	m/s	speed, velocity
L/T^2	m/s^2	acceleration
ML/T	kg m/s	momentum
ML/T^2	kg m/s^2	force
ML^2/T^2	kg m^2/s^2	energy

In speaking of units and dimensions we employ the equivalence symbol: $\equiv$. For example, we might write

$$1 \text{ N} \equiv 1 \text{ kg m/s}^2,$$

which reads "1 newton is defined as 1 kilogram-meter per second squared." The equivalence sign merely signifies that this is a definition, so you can stop worrying about why it is correct. It is also expected that you know how to convert from the CGS system (centimeter-gram-second) to the MKS system.

All the above quantities are extremely important. **Memorize them!**

Example 1.1: Units conversion. A furlong is an obsolete English system unit equal to 201.2 m. Convert furlongs per fortnight to meters per second.

A fortnight is two weeks. Thus, 1 fortnight equals $2 \times 7 \times 24 \times 60 \times 60 = 1.2 \times 10^6$ s. (**Exercise**: convince yourself of this number.). A furlong per fortnight is therefore $202.2/(1.2 \times 10^6) = 1.67 \times 10^{-4}$ m/s, or 0.167 mm/s, slower than a snail's pace, perhaps a molasses's drip.

Note that one day is a little less than 10^5 s, a good number for your number bank.

3. Scalars

All quantities in elementary physics naturally fall into one of two categories that have very different properties: **scalars** and **vectors**. Because they have different properties, they must be treated differently.

A **scalar** is a quantity that can be described by specifying a single number: its **magnitude**, or size. Common examples of scalars encountered in College Board problems are: distance, speed, mass, temperature, density, height, energy, time. Once you have given a scalar's magnitude, you have said all there is to say about it. "The car is moving at 90 kilometers per hour," or "The car is moving at 10 meters per second," fully specifies the speed of the car. Scalars are usually displayed in italics: $v = 90$ km/h.

Scalar and vector addition crop up everywhere in physics and so it is important to understand how to do this properly. For scalars it is easy.

Scalar addition: Two scalars are added by adding their magnitude (there is nothing else to add). The total height above ground of a 2-meter-tall man standing on a 50-meter-tall building is 52 meters. A 5 kg mass added to a 6 kg mass gives an 11 kg mass.

4. Vectors

A **vector** is a quantity that must be described by specifying two properties: a **magnitude** and a **direction**. Common examples of vectors are: displacement, velocity, acceleration, momentum, force. "The car is moving 60 miles per hour at a heading of 90°," gives the car's velocity, as opposed to speed, which is the magnitude of the velocity. The best, and most common, way to visualize a vector is as an arrow:

The arrow's length gives the vector's magnitude and the tip points in the vector's direction. Vectors are generally displayed by either bold face type, such as **v**, or by putting an arrow over the quantity: $\vec{v}$. We will use the former throughout this book. The magnitude, or length. of a vector **v** is designated by |**v**| or just v. For the above car, we could write $v = |\mathbf{v}| = 60$ mph. *The magnitude of a vector is a scalar*.

Note: When you move a vector parallel to itself, neither its length nor direction changes. Thus, it remains the same vector.

Warning: the length of a vector is usually *not* physical length; its length has the dimensions and units of the quantity it represents, for example, mph or newtons.

Every time you add two forces or momenta together, you are adding vectors. Indeed, these quantities naturally have a direction. This is why knowing how to add vectors is crucial. The rules for vector addition are much more complicated than for scalar addition. The first thing to remember is that scalars can be added only to scalars and vectors can be added only to vectors. You cannot add a scalar to a vector. They are apples and oranges.

Vector addition: When adding vectors, their directions must be taken into account. Vectors in opposite directions tend to "cancel each other out."

Tip: Two vectors of equal magnitude but opposite direction sum to zero.

In general, two vectors **A** and **B** are added by placing them "head to tail" as in Figure 1.1. The **resultant vector** **C** = **A** + **B** is by definition the vector drawn from the tail of **A** to the tip of **B**.
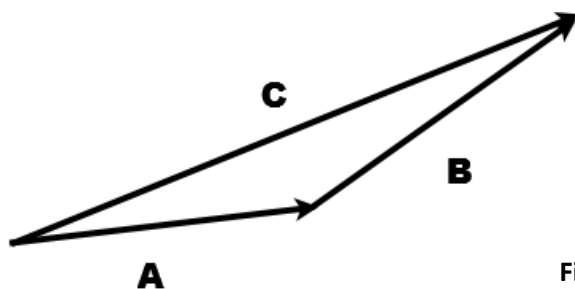
Figure 1.1

The negative of a vector is given by rotating it $180°$. Also, $\mathbf{C} = \mathbf{A} - \mathbf{B}$ is by definition given by

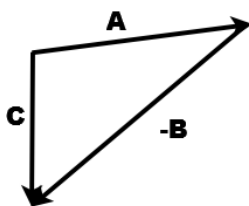$$\mathbf{C} = \mathbf{A} + (-\mathbf{B}):$$

Figure 1.2

One almost always wants to know the magnitude of the resultant, $|\mathbf{C}|$. To find $|\mathbf{C}|$ you can draw it, as above, and measure its length. This gives an approximate answer. To find the exact magnitude of a resultant, the general prescription is:

1. Find the x- and y-components of $\mathbf{A}$ and $\mathbf{B}$. ("x-component" refers to the part of the vector lying in the x-direction. In three dimensions, there are x-, y- and z-components.)

2. Add the x-components of $\mathbf{A}$ to the x-components of $\mathbf{B}$. Similarly, for the y-components (and z-components if there is a third dimension).

3. Square the sums and add them.

4. Take the square root.

Essentially, **one uses the Pythagorean theorem to find the magnitude of the resultant.**

Example 1.2: Adding vectors.
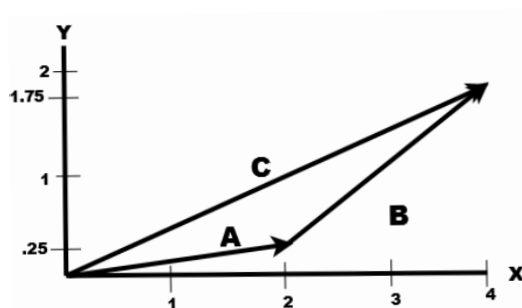
Figure 1.3

Here, the vector $\mathbf{A}$ has x-component 2 and y-component 0.25. Vector $\mathbf{B}$ has x-component 2 and y-component 1.5. (Why? Think parallel.) Thus, vector $\mathbf{C}$ has x-component $2 + 2 = 4$ and y-component $.25 + 1.5 = 1.75$. The magnitude of $\mathbf{C}$ is $|\mathbf{C}| = \sqrt{4^2 + 1.75^2} = 4.37$.

This example illustrates several important properties of vectors:

The magnitude of $C = A + B$ is generally NOT the magnitude of A plus the magnitude of B.

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You must always add x-components to x-components and y-components to y-components.

The maximum possible magnitude of $\mathbf{C} = \mathbf{A} + \mathbf{B}$ occurs when $\mathbf{A}$ and $\mathbf{B}$ are pointing in the same direction. Then $|\mathbf{C}| = |\mathbf{A}| + |\mathbf{B}|$. The minimum possible magnitude of $\mathbf{C}$ will be when $\mathbf{A}$ and $\mathbf{B}$ are pointed in opposite directions. Then $|\mathbf{C}| = |\mathbf{A} - \mathbf{B}|$.

5. Unit Vectors

As the name implies, a **unit vector** is a vector with magnitude 1. Unit vectors are important for several reasons. In the first place, since they have unit magnitude, they can be considered "pure directions" or "direction pointers." Furthermore, any arbitrary vector can be decomposed into unit vectors, in particular, the unit vectors lying along the x-, y- and z-axes. The x-, y- and z- unit vectors are usually written $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ (read "i-hat, j-hat and k-hat") and are illustrated below.
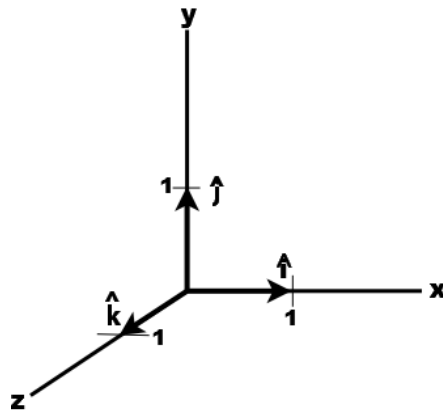
Figure 1.4

A vector of length 6 lying along the x-axis may thus be written as $\mathbf{A} = 6\hat{\mathbf{i}}$, while a vector of length 7 lying along the y-axis may be written as $\mathbf{B} = 7\hat{\mathbf{j}}$. The 6 and 7 represent the magnitudes and $\hat{\mathbf{i}}$ and $\hat{\mathbf{j}}$ give the directions. In Example 1.2 above, we could have written $\mathbf{A} = 2\hat{\mathbf{i}} + 0.25\hat{\mathbf{j}}$ and $\mathbf{B} = 2\hat{\mathbf{i}} + 1.5\hat{\mathbf{j}}$. In doing this we have *resolved* $\mathbf{A}$ and $\mathbf{B}$ into x- and y-components. Then $\mathbf{C}$ is obtained by adding the components separately: $\mathbf{C} = 4\hat{\mathbf{i}} + 1.75\hat{\mathbf{j}}$, and $|\mathbf{C}|$ is found by squaring the components separately, adding them and taking the square root: $|\mathbf{C}| = \sqrt{4^2 + 1.75^2} = 4.37$, as we already determined.

Whenever you are presented with a vector, you should *immediately* think of resolving it into components. This is especially necessary in force and momentum problems.

Given any vector in any direction, one can always convert it into a unit vector by dividing the vector by its magnitude.

Example 1.3: Forming unit vectors. Let $\mathbf{A} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$. Then the magnitude of $\mathbf{A}$ is $\sqrt{13}$. Therefore, $\hat{\mathbf{A}} = \frac{1}{\sqrt{13}}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$ is a unit vector pointing in the direction of $\mathbf{A}$.

6. Scalar Products

Just as one can add vectors, one can also multiply them. However, while it is clear that $3 \times 5 = 15$, it is not so clear how to multiply two vectors $\mathbf{A}$ and $\mathbf{B}$ when they both have not only a magnitude (3 and 5) but a direction as well (north and north west). In elementary physics, two types of vector multiplication prove extremely useful.

The **scalar product**, or **dot product**, of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined as

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$$\mathbf{A} \cdot \mathbf{B} \equiv AB \cos \theta \quad (1.1)$$

where A and B are the magnitudes of $\mathbf{A}$ and $\mathbf{B}$, and θ is the angle between them (Figure 1.5). It is called the scalar product because, as you see from the definition, the result is a scalar rather than a vector.
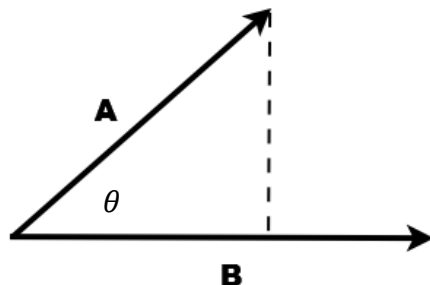

Figure 1.5

Note several important properties of the dot product. First, from the definition you can see that the dot product is *commutative*. In other words, $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$. You should also check that the dot product is *distributive* over addition. That is, $\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$.

Second, the definition implies that for any two parallel vectors, $\mathbf{A} \cdot \mathbf{B} = AB \cos(0) = AB$. In other words, $\mathbf{A} \cdot \mathbf{B}$ becomes just the product of their lengths. In particular, $\mathbf{A} \cdot \mathbf{A} = A^2$. This last fact is extremely valuable: **dotting a vector into itself gives the square of the magnitude.**

The previous statement applies equally to unit vectors. That is, $\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}|^2 = 1 = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}$. Similarly, because $\hat{\mathbf{i}}, \hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are all at right angles to one another, $\theta = 90°$, which implies that $\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$.

Let us write two arbitrary vectors as $\mathbf{A} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}}$ and $\mathbf{B} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}}$, where the subscripts x and y refer to the x- and y-components of the vectors (see Example 1.2 above). Then, taking the dot product, gives

$$\mathbf{A} \cdot \mathbf{B} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}}) \cdot (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}}) = A_x B_x \, \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} + A_x B_y \, \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} + A_y B_x \, \hat{\mathbf{j}} \cdot \hat{\mathbf{i}} + A_y B_y \, \hat{\mathbf{j}} \cdot \hat{\mathbf{j}}$$
$$= A_x B_x + A_y B_y$$

This is often regarded as an alternative definition of the dot product:

$$\mathbf{A} \cdot \mathbf{B} = A_x B_x + A_y B_y \quad (1.2)$$

Note that $\mathbf{A} \cdot \mathbf{A} = A_x^2 + A_y^2 = A^2$ by the Pythagorean theorem. (**Exercise:** Show that definition (1.2) is equivalent to definition (1.1)).

Finally, referring to Figure 1.5, because $A \cos \theta$ is the length of $\mathbf{A}$ that lies along $\mathbf{B}$, we can also write

$$\mathbf{A} \cdot \mathbf{B} = B \times (\text{the projection of } A \text{ on } B).$$

Also convince yourself that

$$\mathbf{A} \cdot \mathbf{B} = A \times (\text{the projection of } B \text{ on } A).$$

Note that in the above two formulas, the symbol $\times$ is being used for ordinary multiplication of scalars. This shouldn't be confused with the cross product, a multiplication between two vectors, which we define below.

Example 1.4: Angles between vectors. One of the most common applications of the dot product is to find the angles between vectors. From definition Eq. (1.1), as long as $\mathbf{A}$ and $\mathbf{B}$ are not the zero vector,

$$\frac{\mathbf{A}}{A} \cdot \frac{\mathbf{B}}{B} = cos\theta \ .$$

But $\mathbf{A}/A$ is by definition $\widehat{\mathbf{A}}$, the unit vector in the direction of $\mathbf{A}$, with a similar expression for $\widehat{\mathbf{B}}$. Thus, to find the angle between two vectors $\mathbf{A}$ and $\mathbf{B}$, you merely need to form the unit vectors in their direction, take the dot product and then the inverse cosine. If $\mathbf{A} = 2\hat{\imath} + 4\hat{\jmath}$ and $\mathbf{B} = 3\hat{\imath} - 2\hat{\jmath}$, we have $A = 2\sqrt{5}$ and $B = \sqrt{13}$. Then

$$\widehat{\mathbf{A}} \cdot \widehat{\mathbf{B}} = \frac{1}{2\sqrt{65}}(6 - 8) = \cos\theta,$$

or $\cos\theta = -0.124$. Taking the inverse cosine gives $\theta = 97.1$. (**Exercise:** Accurately draw the vectors and confirm the calculation visually.)

7. Vector Products

The **vector product**, or **cross product**, of two vectors $\mathbf{A}$ and $\mathbf{B}$ is defined, in contrast to the dot product, as a vector $\mathbf{C}$. The standard notation is $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ (read "C = A cross B"). If $\mathbf{C}$ is the cross product of $\mathbf{A}$ and $\mathbf{B}$, then

$$C \equiv AB \sin\theta \quad (1.3)$$

C points in the direction given by the right-hand rule.

Here, only the *magnitude* of $\mathbf{C} = C$ is given by $AB \sin\theta$, where, again, θ is the angle between $\mathbf{A}$ and $\mathbf{B}$.

The direction of $\mathbf{C}$ is given by the infamous **right-hand rule (RHR)**, which says: To cross $\mathbf{A}$ into $\mathbf{B}$, point the fingers of your right hand along the direction of $\mathbf{A}$. Curl your fingers in the direction of $\mathbf{B}$. Your thumb will point in the direction of $\mathbf{C}$ (Figure 1.6). Alternatively, you can visualize a screw with a normal right-hand thread sitting at the junction of $\mathbf{A}$ and $\mathbf{B}$. If you imagine inserting a screwdriver into the screw and turning it so that $\mathbf{A}$ rotates into $\mathbf{B}$, the screw will advance in the direction of $\mathbf{C}$.
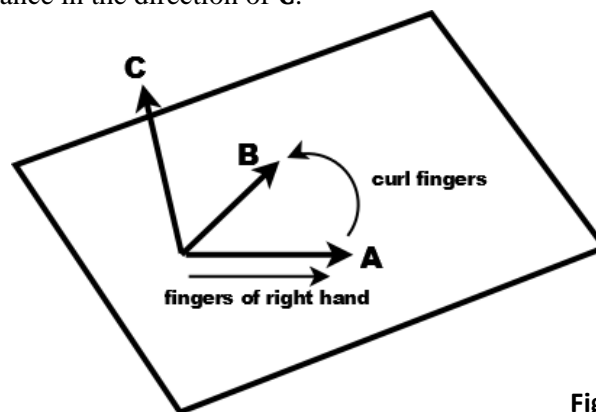
Figure 1.6

Note: C is at right angles to both A and B.

Warning: To get the right answer by the right-hand rule, you must use your right hand.

The warning is another way of stating that $\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$. *The cross product is not commutative*.

Although the cross product initially seems strange, it turns out to be extremely useful because in certain situations involving torques and magnetic fields (see Sections 4.6 and 9.10) a force may act at right angles to the direction of an object's motion.

Example 1.5: Ball on string. A ball is attached to the ceiling and is whirling around on a string, as shown. If $\boldsymbol{\ell}$ is the vector pointing from the point of attachment to the ball, and $\mathbf{v}$ is the vector pointing in the direction of the ball's motion (the velocity), in what direction does $\mathbf{v} \times \boldsymbol{\ell}$ point?
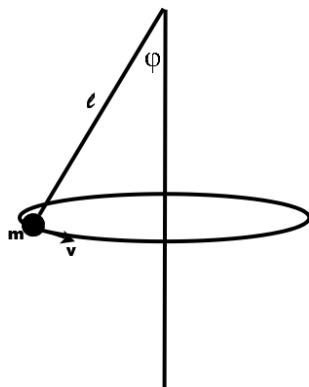
Using the RHR, the resultant points up and to the left, at an angle φ from the horizontal. (**Exercise:** Convince yourself of the last statement.)

PROBLEMS INVOLVING DIMENSIONS, UNITS AND VECTORS

Problems 1-11 are at the subject-test level. The remaining problems are at the AP level.

1. Copper has a density of nearly 9 grams per cubic centimeter. Expressed in terms of kilograms per cubic meter copper's density is:

   (A) 0.9 kg/m^3
   (B) 9 kg/m^3
   (C) 90 kg/m^3
   (D) 900 kg/m^3
   (E) 9000 kg/m^3

2. In the weekly lab, students were asked to weigh their physics texts to one percent. Which of the following would be the likely mass they measured?

   (A) 0.238 kg
   (B) 3.2 kg
   (C) 3.44 kg
   (D) 10.7 kg
   (E) 50 kg

3. Which of the following are vector quantities?

   a) Velocity   b) Speed   c) Displacement   d) Mass   e) Acceleration

   (A) a
   (B) a, b, e
   (C) a, d
   (D) a, c, e
   (E) b, d

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4. If a vector **A** of length 5 is added to a vector **B** of length 4, the resultant vector will:

    (A) have a magnitude of 9.
    (B) have a magnitude of 1.
    (C) have a magnitude of −1.
    (D) have a magnitude somewhere between 0 and 9, inclusive.
    (E) have a magnitude somewhere between 1 and 9, inclusive.

5. A cyclist rides a bike 6 kilometers north and 9 kilometers east. Which is nearest to the final distance from her starting point?

    (A) 3 km
    (B) 9 km
    (C) 10 km
    (D) 16 km
    (E) 20 km

6. The same cyclist continues her trip by riding south for a distance of 11 km. Which of the below vectors best illustrates her final position from her starting point. Each tick mark represents two km; to the right is east and up is north.

    (A) A
    (B) B
    (C) C
    (D) D
    (E) E
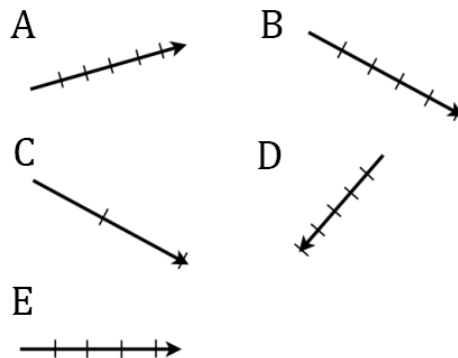
7. Pressure is defined as a force per unit area. Which of the following has the same dimensions as pressure?

    (A) Energy per unit volume
    (B) Momentum per unit area
    (C) Momentum per unit length
    (D) Acceleration per unit area
    (E) Momentum per unit volume

8. A pendulum has a length ℓ, a mass m and is swinging under the action of Earth's gravity, which produces an acceleration g. Expressed in terms of these quantities, the pendulum's frequency, in cycles per second, could be:

    (A) $\sqrt{mg/\ell}$
    (B) $\sqrt{\ell/g}$
    (C) g/ℓ
    (D) $\sqrt{\ell g/m}$
    (E) $\sqrt{g/\ell}$

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9. Power is defined as energy per unit time. Which of the following also represents power?

    (A) Momentum times acceleration
    (B) Momentum times velocity
    (C) Force times distance
    (D) Force times velocity
    (E) Both (A) and (D)

10. A sphere of mass M and initial radius R is compressed so that its final radius is $R/2$. How does the density change?

    (A) It stays the same.
    (B) It decreases by a factor of two.
    (C) It increases by a factor of two.
    (D) It increases by a factor of eight.
    (E) It increases by a factor of four.

Questions 11 - 12 refer to the following figure.
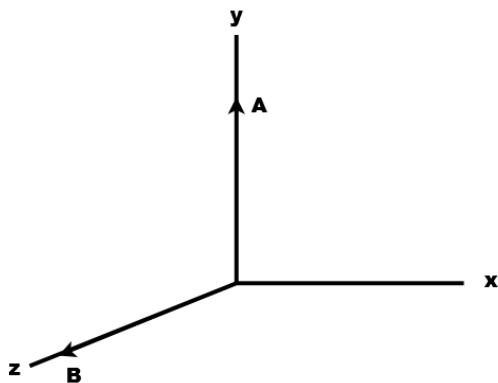
11. If **A** is a vector pointing along the y-axis and **B** is a vector pointing along the $+z$-axis, as shown above, then the vector $\mathbf{A} \times \mathbf{B}$

    (A) points along the $+y$-axis.
    (B) points along the $+x$-axis.
    (C) points along the $+z$-axis.
    (D) points along the $-x$-axis.
    (E) points along the $-z$-axis.

12. If **A** has length 2 and **B** has length 3, then $\mathbf{A} \times \mathbf{B}$ has length

    (A) 1
    (B) 2
    (C) 4
    (D) 6
    (E) 8

13. **A** is a vector with magnitude 3 pointing along the y-axis. The vector **B** has magnitude 4, lies in the xz-plane and points at 45° between the $+z$-axis and the $+x$-axis, as shown below.

The vector **B** × **A**

(A) has magnitude 12, lies in the xz-plane,
and points at 45° from the $+z$-axis to the $-x$-axis.
(B) has magnitude 6, lies in the xz-plane,
and points at 45° from the $+z$-axis to the $-x$-axis.
(C) has magnitude 6, lies in the xz-plane,
and points at 45° from the $+x$-axis to the $-z$-axis.
(D) has magnitude 12, lies in the xz-plane,
and points at 45° from the $+x$-axis to the $-z$-axis.
(E) has magnitude 12, lies in the xz-plane,
and points at 45° from the $+z$-axis to the $+x$-axis.
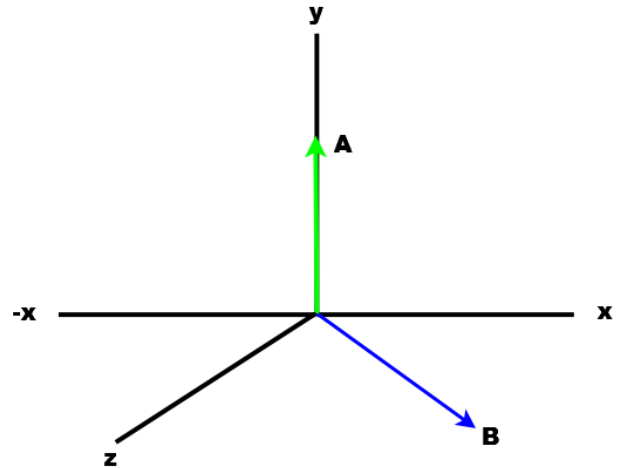
14. **A** is a vector with magnitude 3 pointing along the y-axis. The vector **B** has magnitude 4, points upwards at 60° from the xz-plane at an angle of 60° from the $+z$-axis, as shown, below.

The vector **B** × **A**

(A) has magnitude 6, lies in the xz-plane,
pointing at 60° from the $+x$-axis to the $-z$-axis.
(B) has magnitude 12, lies in the xz-plane,
pointing at 30° from the $+z$-axis to the $-x$-axis.
(C) has magnitude 6, lies in the xz-plane,
pointing at 30° from the $+z$-axis to the $-x$-axis.
(D) has a magnitude 12, lies in the xz-plane,
pointing at 60° from the $+x$-axis to the $-z$-axis.
(E) has a magnitude 6, lies in the xz-plane,
pointing at 60° from the $+z$-axis to the $-x$-axis.
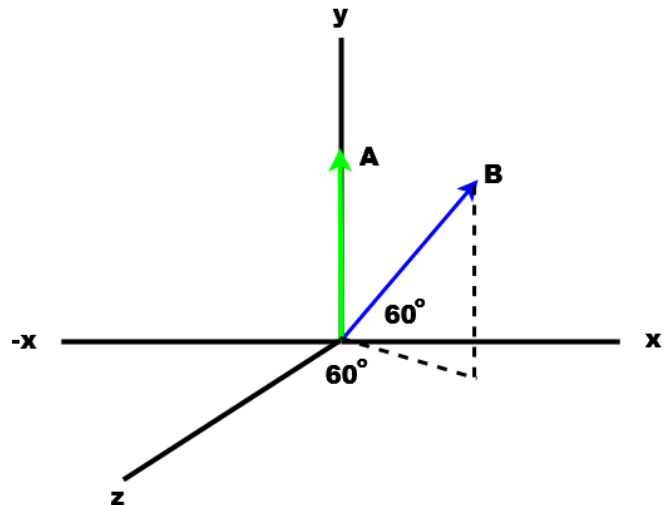
15. If **A** and **B** are vectors, then **B** · (**B** × **A**) is equal to:

(A) **A**
(B) **B**
(C) A^2B
(D) zero
(E) B^2A

16. If **A** and **B** are vectors with an angle less than 90° between them, then we can say:

(A) $|\mathbf{A} + \mathbf{B}| = |\mathbf{A}| + |\mathbf{B}|$
(B) $|\mathbf{A} + \mathbf{B}| = \sqrt{|\mathbf{A}|^2 + |\mathbf{B}|^2}$
(C) $|\mathbf{A} + \mathbf{B}| < \sqrt{|\mathbf{A}|^2 + |\mathbf{B}|^2}$
(D) $|\mathbf{A} + \mathbf{B}| < |\mathbf{A}| + |\mathbf{B}|$
(E) (C) and (D)

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17. If $\mathbf{A} = 3\,\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{B} = 6\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$, then a unit vector in the same direction as $\mathbf{B} - \mathbf{A}$ is:

   (A) $\hat{\mathbf{C}} = 9\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$
   (B) $\hat{\mathbf{C}} = -1/5(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}})$
   (C) $\hat{\mathbf{C}} = 1/5(3\hat{\mathbf{i}} - 4\hat{\mathbf{j}})$
   (D) $\hat{\mathbf{C}} = 1/5(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$
   (E) $\hat{\mathbf{C}} = 1/84(9\hat{\mathbf{i}} - 2\hat{\mathbf{j}})$

18. If $\mathbf{A} = \frac{1}{5}(3\hat{\mathbf{i}} + 4\hat{\mathbf{j}})$ and $\mathbf{B} = \frac{1}{13}(5\hat{\mathbf{i}} + 12\hat{\mathbf{j}})$, then the angle between these two vectors is nearest

   (A) 10.3°
   (B) 14.3°
   (C) 15.8°
   (D) 25.7°
   (E) 30.4°

2. KINEMATICS REVIEW

This entire chapter is at the subject-test level, although some problems are at the AP level.

1. Essential Terms

Mechanics is the branch of physics that deals with the motion of material objects under the action of **forces**. The term **kinematics**, which literally means "the study of motion," is the branch of mechanics in which the forces causing the motion are ignored; we merely consider the motion itself, not the causes.

The basic quantities under consideration in kinematics problems are **displacement**, **velocity** and **acceleration**. They are all **vector** quantities and must be treated as such.

2. Displacement and Velocity

Displacement is defined as the *position of an object from its point of origin after moving a time t.* Precisely, *displacement is the **vector** that points from the point of origin to the position of an object after a time t* (See Figure 2.1). The route the object took to get there is not important. People often speak of displacement and distance as being the same, but they are not. In making a round trip from New York to Philadelphia, you traverse a *distance* of about 300 kilometers, but your total *displacement* is zero. The dimensions of displacement and distance are length, L, and the SI unit of both is the meter, m. Displacement is often indicated by the letter **s**. Distance is a scalar and is usually indicated by Latin letters in italics, such as s or x.
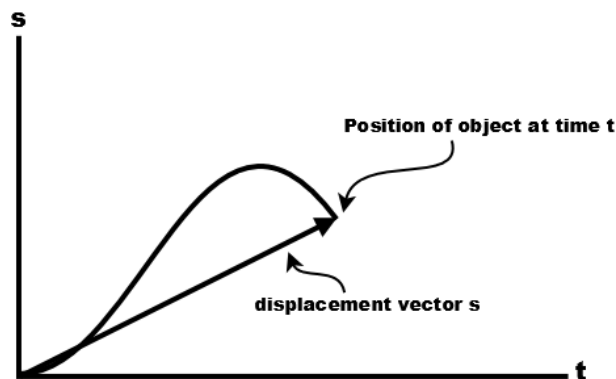
Figure 2.1

Velocity is defined as the *change in displacement per unit time*, or *the rate of change of the displacement vector*. **Speed** is the *magnitude of the velocity*, or the length of the velocity vector; speed is a scalar. The dimensions of velocity and speed are L/T. In SI units, velocity and speed are measured in meters per second, m/s.

Recall that the *slope* of a graph (rise over run) of any function represents the rate of change of that function. Also recall that the slope of a graph at any point is by definition the slope of the tangent line to the graph at that point.

> **Supreme Hint:**
>
> **The slope of an object's displacement-versus-time graph at any point (the slope of the tangent at that point) gives the velocity of the object.**

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Commentary on hint: In general, the displacement may be a complicated function of time, as in Figure 2.2.
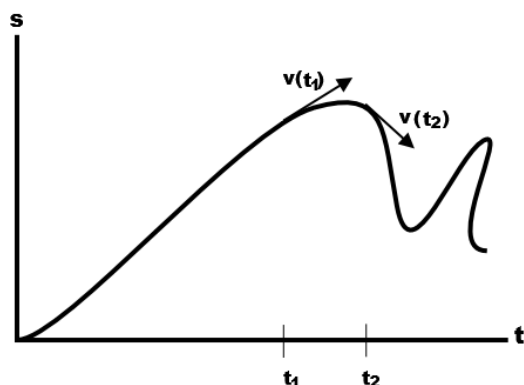
Figure 2.2

Because the slope of the graph changes continually with time *t*, the velocity must change at every point. To precisely find the velocity at each time requires calculus,[1] but over small time intervals we can approximate the velocity by drawing short tangent lines along the curve. Putting an arrowhead on a tangent line makes it a vector. Two such velocity vectors are shown in Figure 2.2. *The magnitude of the slope of the tangent vector gives the speed*. We calculate the slope, as always, by finding the rise over the run. The *rise* is $\mathbf{s}_f - \mathbf{s}_o$, the difference between the final and initial displacements (see figure 2.3, below). The *run* is $t_f - t_o$, the difference between the final and initial times. The velocity is

$$\mathbf{v} \equiv \frac{\mathbf{s}_f - \mathbf{s}_o}{t_f - t_o} = \frac{\Delta \mathbf{s}}{\Delta t} \quad (2.1)$$

The last expression above is read as "Delta $\mathbf{s}$ over Delta t", where the Greek letter Δ almost always stands for a small change in something. (**Exercise**: On Figure 2.2 draw two displacement vectors $\mathbf{s}_1$ and $\mathbf{s}_2$ that give one of the indicated velocity vectors.)

We have written the velocity as a vector, as one should. **In most College Board problems, motion is in one dimension. In that case, there are only two directions, positive and negative. We can then write v as a scalar and indicate $+v$ for velocity in the positive direction and $-v$ for velocity in the negative direction.**

As a one-dimensional example, suppose that an object's displacement $\mathbf{s}$ increases by 5 meters in the positive x-direction over a period of 2 seconds. We would say that the approximate velocity during the interval is $v = +(5/2)$ m/s. Precisely, this is the *average velocity*. We return to this point in Section 3.

Example 2.1: Finding approximate velocity from a displacement-vs.-time graph.

Take the displacement-vs.-time graph in Figure 2.2. We approximate the first hump (roughly) by straight-line segments over small intervals of time and draw Figure 2.3 below.

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[1] Students who have taken calculus know that the instantaneous velocity, $\mathbf{v}(t) \equiv d\mathbf{s}/dt$, is the derivative of the displacement with respect to time.
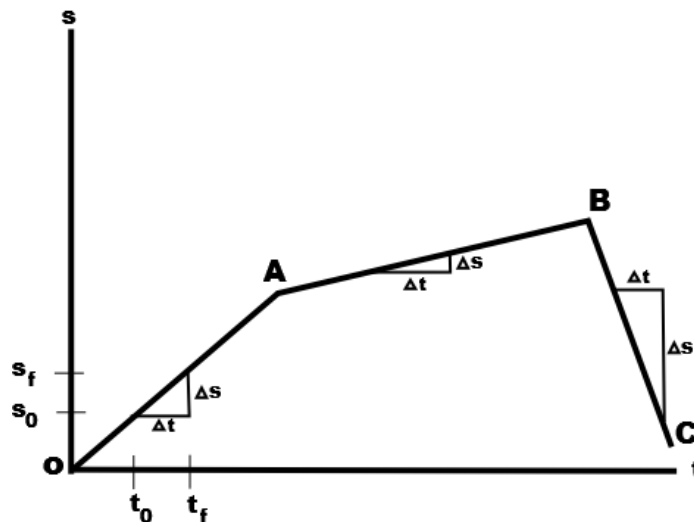
Figure 2.3

These lines segments approximate the tangents to the curve. In the interval **OA**, the slope $\Delta\mathbf{s}/\Delta t$ is positive, so the velocity **v** as defined by Eq. (2.1) is positive and the object is moving in the positive direction. In the interval **AB,** the slope $\Delta\mathbf{s}/\Delta t$ is smaller and positive, so the speed $v = |\mathbf{v}|$ is less but the object still has positive velocity and it continues moving in the positive direction. In the interval **BC**, the slope $\Delta\mathbf{s}/\Delta t$ is very large and negative, so $|\mathbf{v}|$ is very large but the object is moving in the opposite direction. The displacement, however, remains always positive. If point **A** has a displacement of 1 meter at a time of 2 seconds, then the velocity $\Delta\mathbf{s}/\Delta t$ in the interval **OA** is 0.5 m/s in the positive direction.

3. Relationship Between Total Displacement and Velocity

We have defined velocity as the *change* of displacement per unit time, $\mathbf{v} = \Delta\mathbf{s}/\Delta t$, which implies that a *small change* in displacement is $\Delta\mathbf{s} = \mathbf{v}\Delta t$. This statement applies only to small time intervals; it is a *common mistake* to say that the *total* displacement is $\mathbf{s} = \mathbf{v}t$ over a long period of time. This is strictly only true if the velocity is *constant*. Over an extended time interval, however, the velocity may be changing, just as when driving from New York to Philadelphia you may stop for gas or exceed the speed limit. Nevertheless, if someone asked you how fast you drove, you would say, "We drove 150 km in two hours, so we averaged 75 km per hour."

The colloquial statement that "velocity is displacement divided by time" is only approximately true. It is true, however, that the *average velocity*, $\bar{\mathbf{v}}$, is displacement/time, or $\mathbf{s} = \bar{\mathbf{v}}t$. Average velocity is usually written $\bar{\mathbf{v}}$ or $\langle\mathbf{v}\rangle$.

Here again, it is important to distinguish displacement from distance. If you made a round-trip journey from New York to Philadelphia in five hours, the average *velocity* would be zero, because you haven't gone anywhere—the total displacement is zero. On the other hand, the average *speed* would be $300/5 = 60$ km/hr. That is, the average speed is $\overline{v} = \overline{|\mathbf{v}|} =$ (total distance)/time.

Not coincidentally, there is an important graphical relationship between velocity and total displacement. Let us plot the velocities from Figure 2.3. Because the velocity is constant in each segment, one gets something like this:

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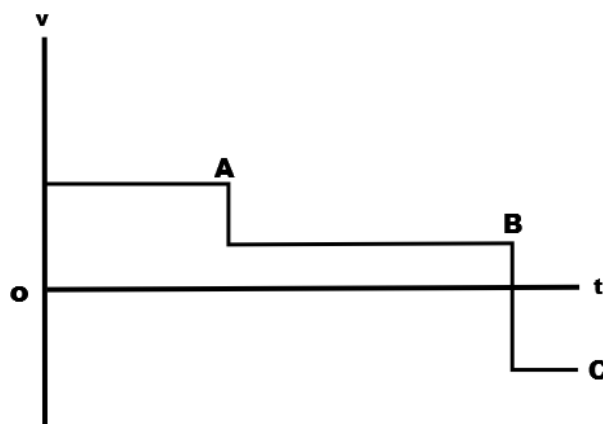
Figure 2.4

Notice that the total displacement, **s**, is the area on a velocity-vs.-time graph (the dimensions of displacement are $[\mathbf{s}]$ = velocity × time). If we calculated the area under the velocity curve in Figure 2.4, taking into account that the area in interval **BC** is counted as negative because it lies beneath the time axis, then the total area would equal the total displacement.

> **The area under a velocity vs. time graph is the total displacement.**

Example 2.2: Finding total displacement from a velocity-vs.-time graph.
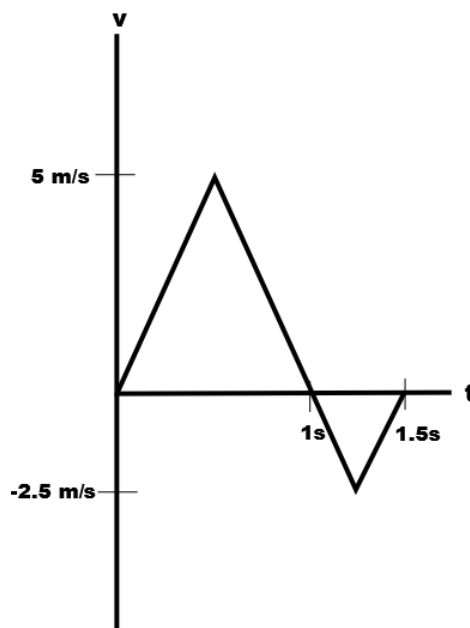
Figure 2.5

The area under the first triangle is $\frac{1}{2} \times 5 = 2.5$ m and the (signed) area under the second triangle is $\frac{1}{2} \times 0.5 \times (-2.5) = -0.625$ m. Adding them shows that the total displacement is $+1.875$ m.

The average velocity $\overline{\mathbf{v}}$ is in fact the velocity drawn on a velocity-vs.-time graph such that the area between it and the time axis equals the area beneath the actual velocity curve. In Figure 2.5, $\overline{v} = 1.25$ m/s because $(1.25 \text{ m/s}) \times (1.5 \text{ s}) = 1.875$ m.

Exercise: Verify the last statement and draw in $\overline{v}$ on Figure 2.5.

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4. Acceleration

Acceleration is defined as the *change in velocity per unit time*, or *acceleration is the rate of change of velocity*. The dimensions of acceleration are L/T^2. Since in the SI system, velocity is measured in m/s, acceleration is measured by meters per second per second, m/s^2.

As with velocity, the precise definition of acceleration at a given instant involves calculus.[2] However, in analogy to the definition of velocity, over small time intervals we can say

$$\mathbf{a} \equiv \frac{\Delta \mathbf{v}}{\Delta t} = \frac{(\mathbf{v}_f - \mathbf{v}_o)}{(t_f - t_o)} \quad (2.2)$$

where $\mathbf{v}_f$ is the final velocity, $\mathbf{v}_o$ is the initial velocity, and as before t_f is the final time and t_o is the initial time. If the velocity $\mathbf{v}$ changes from $\mathbf{v} = 5/2$ m/s in the x-direction to $\mathbf{v} = 9/2$ meters per second in 2 seconds, the acceleration in that interval is $\mathbf{a} = \frac{9/2 - 5/2}{2}$ or $1 \, m/s^2$ in the x-direction. In analogy with the discussion in Section 3, this is technically the *average acceleration*. Also, in analogy with the discussion in Section 3, the area under an acceleration-vs.-time graph is the change in velocity.

Example 2.3: What is the acceleration in Figure 2.5?

The slope of the velocity is constant in each time interval in Figure 2.5, and so the acceleration is as well. The initial acceleration $\mathbf{a} = 5/0zx. 5 = 10 \, m/s^2$ in the positive direction. The slope in the next interval, until the apex of the small triangle, has the same magnitude but is negative, so $\mathbf{a} = 10 \, m/s^2$ in the negative direction. The final acceleration is once again $\mathbf{a} = 10 \, m/s^2$ in the positive direction

5. Motion for Constant Acceleration

Just as the proper definition of acceleration in terms of velocity requires calculus, the reverse is also true. If an object's acceleration is changing with time, such that if $\mathbf{a} = \mathbf{a}(t)$, then it may be difficult, if not impossible, to find the exact velocity.[3]

Tip: Virtually all high-school and College Board problems assume that a is constant.

This assumption makes life much easier. For constant $\mathbf{a}$, the definition of acceleration in Eq. (2.2) equals the average acceleration for all time intervals and by cross-multiplying we can write

$$\mathbf{v}_f - \mathbf{v}_o = \mathbf{a}(t_f - t_o) \quad (2.3)$$

Usually we take the initial time to be zero, in which case t_f is just the total time t. Then

$$\mathbf{v}_f = \mathbf{a}t + \mathbf{v}_o \quad (2.4)$$

Thus, for constant acceleration, the velocity is just the acceleration multiplied by the time, as is commonly said, plus any initial velocity. Plotting Eq. (2.4) gives us the following picture.

---

[2] In analogy with velocity, $\mathbf{a}(t) \equiv d\mathbf{v}/dt$; acceleration is the derivative of the velocity with respect to time.
[3] In general, velocity is the integral of the acceleration with respect to time, $\mathbf{v}(t) = \int \mathbf{a}(t)dt$.
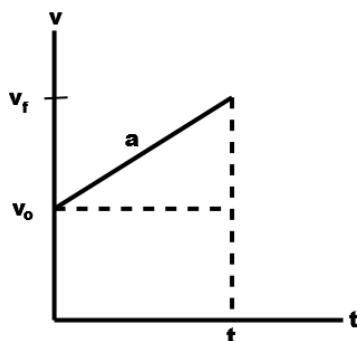
Figure 2.6

The area under this curve is $1/2(\mathbf{v}_f - \mathbf{v}_o)t + \mathbf{v}_o t = 1/2(\mathbf{v}_o + \mathbf{v}_f)t$. From Section 3 we know that this area equals the total displacement $\mathbf{s} = \bar{\mathbf{v}}t$, which implies that the object's average velocity is just $\bar{\mathbf{v}} = (\mathbf{v}_f + \mathbf{v}_o)/2$.

Using Eq. (2.4) for $\mathbf{v}_f$, we have

$$\mathbf{s} = \bar{\mathbf{v}}t = \left[\frac{\mathbf{v}_f + \mathbf{v}_o}{2}\right]t = \left[\frac{\mathbf{a}t + \mathbf{v}_o}{2}\right]t + \frac{\mathbf{v}_o}{2}t ,$$

or

$$\mathbf{s} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_o t \quad (2.5a)$$

Note that the displacement $\mathbf{s}$ is the final position minus the initial position. Thus, if we let $\mathbf{s} = \mathbf{x} - \mathbf{x}_o$ where $\mathbf{x}$ and $\mathbf{x}_o$ represent the final and initial position vectors, respectively, Equation (2.5a) can be written

$$\mathbf{x} = \frac{1}{2}\mathbf{a}t^2 + \mathbf{v}_o t + \mathbf{x}_o \quad (2.5b)$$

This form is often encountered when the initial position is explicitly stated as not equal to zero.

Equations (2.4) and (2.5) are vector equations. If the motion is in only one dimension, we can write $\pm a$, $\pm v$ and $\pm s$ for motion in the positive (negative) direction, as was already discussed. Then, substituting $t = (v_f - v_o)/a$ from Eq. (2.4) into Eq. (2.5a) gives

$$s = \frac{1}{2}\frac{a(v_f - v_o)^2}{a^2} + v_o\frac{v_f - v_o}{a}$$

or, squaring the expression in parentheses and cancelling terms,

$$2as = v_f^2 - v_o^2 \quad (2.6)$$

Eq. (2.6) involves only scalar quantities (distance and speeds). Equations (2.4), (2.5) and (2.6) constitute the basic kinematic equations and are all you need to know to solve virtually any SAT subject question on kinematics. Memorize them**.**

Example 2.4: One–dimensional motion. A ball is thrown straight up into the air. What is the maximum height reached? What is the time required to reach the maximum height?

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This is a simple application of Eqs. (2.4) and (2.5) and is used all the time. Once the ball leaves the hand, the only acceleration is that due to the Earth's gravity, $g = 9.8$ m/s^2 acting in the negative (downward) direction. The maximum height y_m reached by the ball is attained when the acceleration due to gravity brings the velocity to zero. Thus, using Eq. (2.4), we get $0 = v_o - gt$. Hence, $t = v_o/g$. Plugging this result into Eq. (2.5) gives for the maximum height $y_m = -\frac{1}{2}g\left(\frac{v_o}{g}\right)^2 + v_o(\frac{v_o}{g})$, or $y_m = \frac{1}{2}v_o^2/g$. If you throw a ball up at 10 m/s this means the time to reach the maximum height is $t = 10/9.8 \approx 1$ s. The maximum height is $y_m = \frac{1}{2}100/9.8 \approx 5$ m.

Exercise: If a ball is thrown upward, show that the time to reach the maximum height is the same as the time to fall from the maximum height to the ground. Thus, the roundtrip time is $t = 2v_o/g$. Show also that the speed with which the ball hits the ground is the same as the speed with which it is thrown upward.

Example 2.5: Two–dimensional motion. A projectile is fired at an angle $\theta = 60°$ from the horizontal at an initial speed $v_o = 50$ m/s, as shown below. What is the horizontal range R attained by the projectile? Ignore any air resistance.
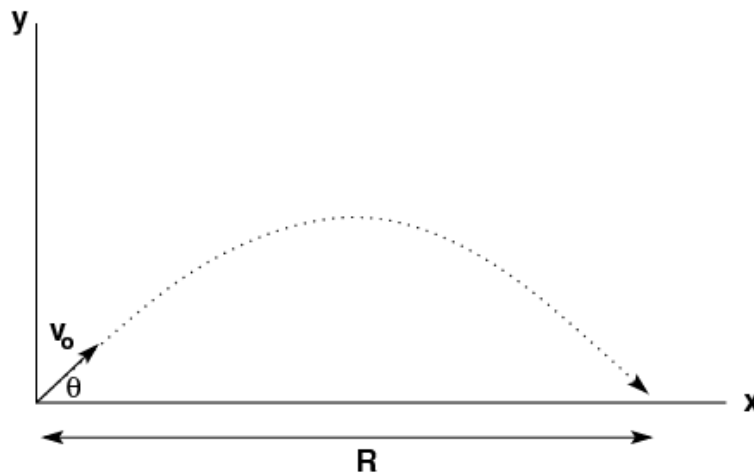
Figure 2.7

This is a basic problem whose result is used in any number of college exam questions. The important point is: *velocity is a vector*. Because the problem is two-dimensional, the velocity has both x- and y-components that can and must be treated separately. We write $v_o = v_{ox}\,\hat{\mathbf{i}} + v_{oy}\,\hat{\mathbf{j}}$, where v_{ox} is the initial speed in the x-direction and v_{oy} is the initial speed in the y-direction. As stressed in Section 1.4, **any time you are presented with a problem in two dimensions you should immediately resolve the vectors into components. The components act independently of one another. You have essentially turned one problem into two separate problems.**

In this example, there is no acceleration in the x-direction, and so the x-velocity remains constant: $v_x = v_{ox}$. The range is then simply $R = v_{ox}t$, where t is the time necessary for the projectile to hit the ground. From Example 2.4 (and the following exercise), $t = 2v_{oy}/g$ because only the velocity in the y-direction determines the projectile's maximum height. Thus, $R = 2v_{ox}v_{oy}/g$. From the figure we see that $v_{ox} = v_o \cos\theta$ and $v_{oy} = v_o \sin\theta$. Therefore, $R = 2v_o^2 \sin\theta \cos\theta/g$. Using the double-angle trig identity, one can rewrite this as $R = v_o^2 \sin 2\theta/g$. For the given numbers, $R = 2500 \times \frac{\sqrt{3}}{2}/9.8 \approx 220$ m. Notice that R is maximized when $\sin 2\theta = 1$. Consequently, we have the important result that, barring air resistance, the maximum range of a projectile is attained by firing it at 45°.

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KINEMATICS PROBLEMS

Questions 1 - 6 refer to the following information.
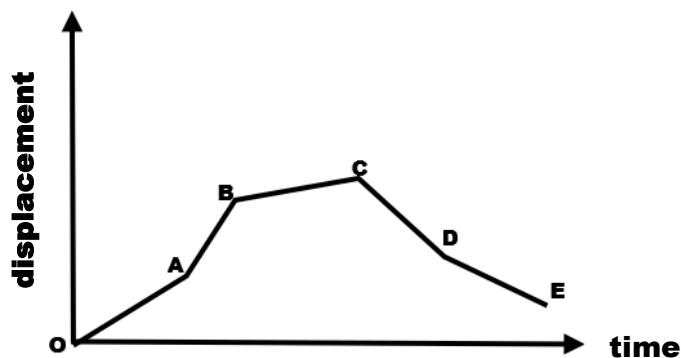
A driver gets into his car at home and drives off along a straight road for a time until he stops for a rest. His initial direction is positive.

1. In which segment is his velocity most positive?

    (A) OA
    (B) AB
    (C) BC
    (D) CD
    (E) DE

2. In which segment is his speed the greatest?

    (A) OA
    (B) AB
    (C) BC
    (D) CD
    (E) DE

3. At which point has he traveled the furthest total distance from his starting point?

    (A) A
    (B) B
    (C) C
    (D) D
    (E) E

4. At which point is he farthest away from his starting point?

    (A) A
    (B) B
    (C) C
    (D) D
    (E) E

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5. In which segment is his velocity the most negative?

    (A) OA
    (B) AB
    (C) BC
    (D) CD
    (E) DE

6. In which segment is his acceleration the least?

    (A) BC
    (B) DE
    (C) There is not enough information to solve the problem.
    (D) None
    (E) All the accelerations are equal.

7. A lady bug walks across a table in a straight line for 4 seconds, crossing 4 cm, then turns around and runs back in the opposite direction for 4 seconds, crossing 8 cm. Assume that the horizontal axes in all the below graphs represent time and that the scales are uniform. Which pair of graphs below could represent the lady bug's displacement and velocity, respectively, with her average velocity drawn in.

    (A) A and B
    (B) A and F
    (C) D and E
    (D) C and F
    (E) C and B
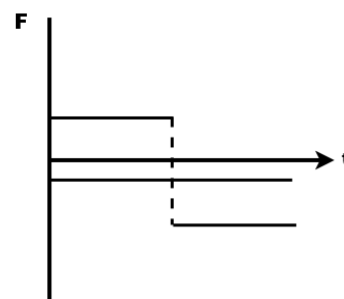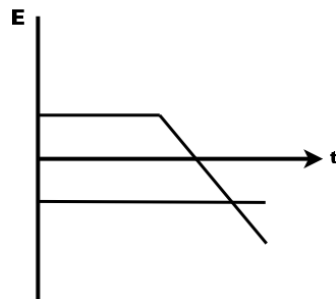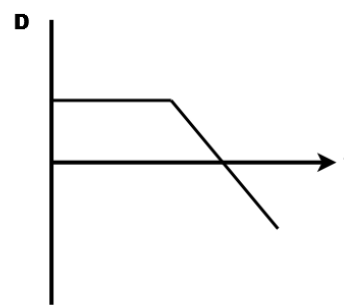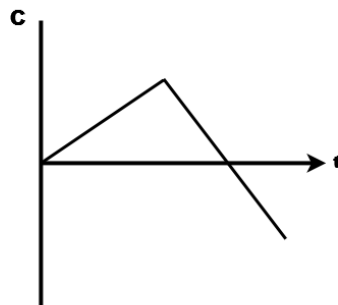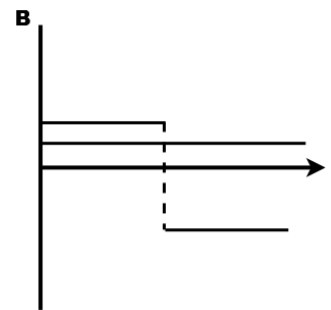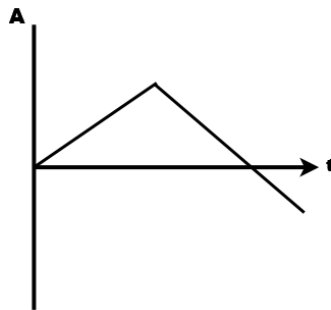
Questions 8 - 9 refer to the following figure.
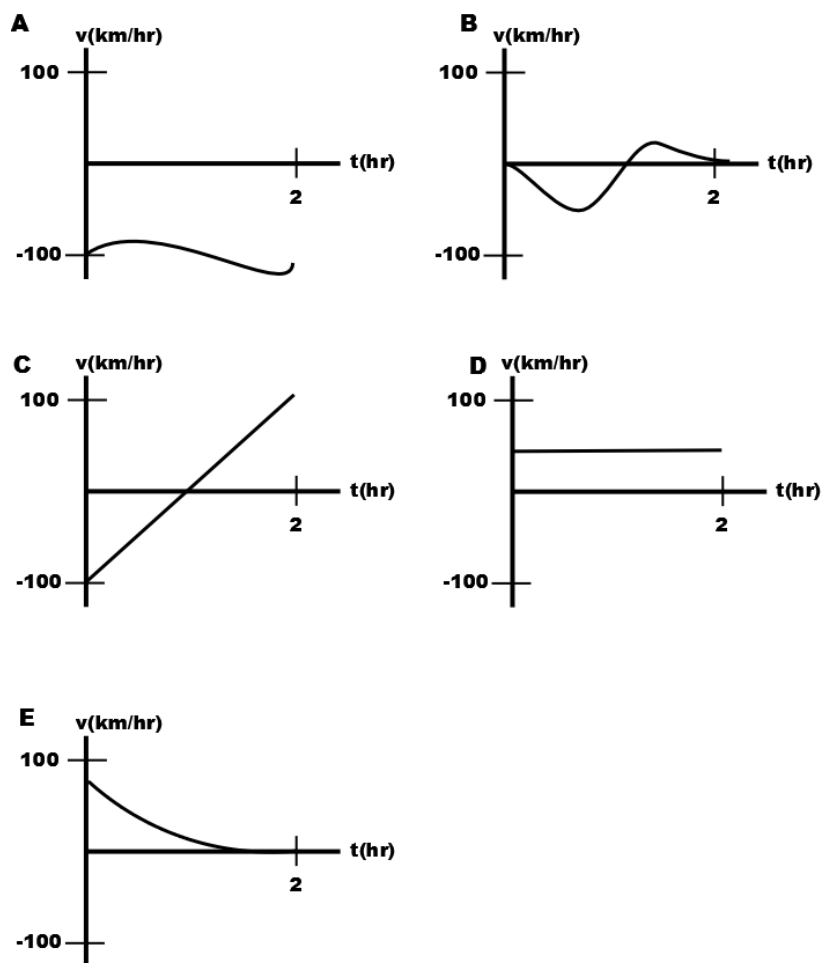
8. Five drivers set off and drive for two hours on five straight roads. At the end of two hours, which car has the greatest displacement from its starting point?

    (A) A
    (B) B
    (C) C
    (D) D
    (E) E

9. Which two graphs have the closest average speeds?

    (A) A and D
    (B) A and C
    (C) C and D
    (D) C and E
    (E) B and D

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10. A model rocket is launched into the air and after several seconds the engine has burned up all its propellant. Thereafter, the rocket continues to rise until it reaches a maximum height, a parachute opens and the rocket descends to the field. Assuming positive is up and negative is down, which are the correct signs of the displacement, velocity and acceleration during the ascent, after engine shutoff?

(A) A
(B) B
(C) C
(D) D
(E) E

	Displacement	Velocity	Acceleration
A	Positive	Positive	Positive
B	Negative	Positive	Positive
C	Positive	Positive	Negative
D	Positive	Negative	Negative
E	Negative	Negative	Negative

11. Consider the same rocket as in the previous problem. What are the correct signs of the displacement, velocity and acceleration during the descent?

(A) A
(B) B
(C) C
(D) D
(E) E

12. A car sets off at time zero due north with constant initial velocity, $v_o > 0$. At time t_o it begins to accelerate due south. Which of the below graphs could represent the car's displacement, with north taken as positive.

(A) A
(B) B
(C) C
(D) D
(E) E

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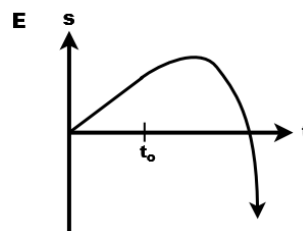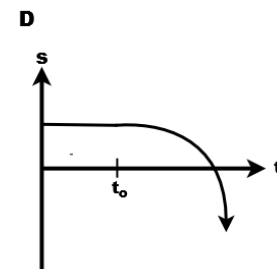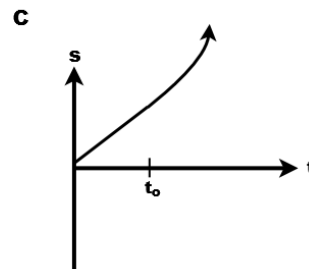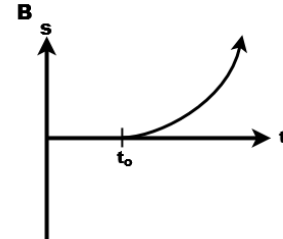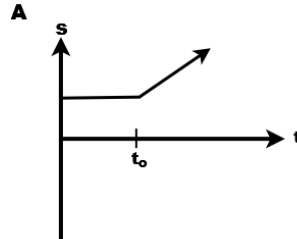
13. A satellite in orbit around the Earth at a distance $R = 6800$ km from the Earth's center, completes 3 complete orbits in 4.5 hours. The satellite's average speed in km/s is:

(A) $\dfrac{2\pi(6800)}{1.5} \dfrac{1}{3600}$

(B) $\dfrac{2\pi(6800)}{4.5} \dfrac{1}{3600}$

(C) $\dfrac{6800(3600)}{2\pi(4.5)}$

(D) $\dfrac{2\pi(3600)}{4.5(6800)}$

(E) $\dfrac{2\pi}{1.5(6800)} \dfrac{1}{3600}$

14. Shown below is a displacement-vs.-time graph for an object undergoing constant acceleration.
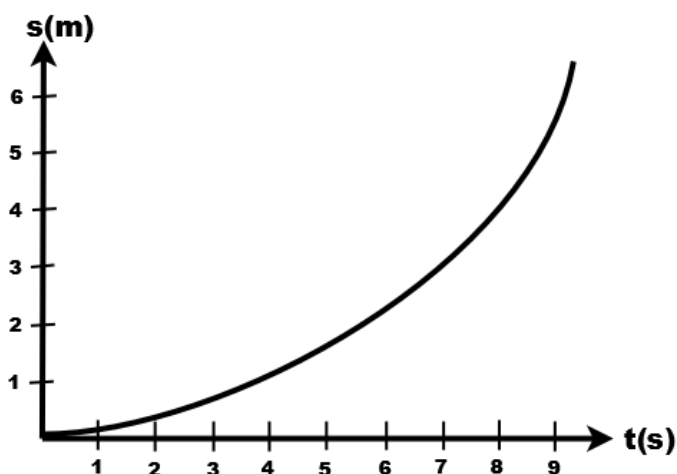
From this graph you can conclude:

(A) that the average velocity between 4 and 6 seconds is approximately 0.5 m/s.
(B) that the average velocity between 7 and 8 seconds is about 1 m/s.
(C) that the acceleration is about $\frac{1}{8}\,\text{ms}^{-2}$.
(D) (A) and (C)
(E) (A), (B) and (C)

15. You wish to cross a river in a boat so that you end up on the opposite bank directly across from your starting point. You are able to row the boat at 3 km/hr in still water and the river is flowing to the left at 1 km/hr. The river is 1 km wide. Your strategy for crossing is best illustrated by which of the below diagrams?

(A) A
(B) B
(C) C
(D) D
(E) E
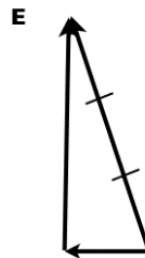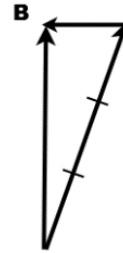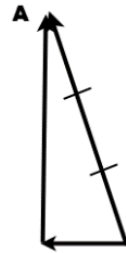
16. How long does it take you to cross the river in problem 15?

(A) 1/4 hr
(B) 1/3 hr
(C) $1/(2\sqrt{2})$ hr
(D) 1 hr
(E) $2\sqrt{2}$ hr

17. During a shower, rain is falling vertically at its terminal velocity v_r. A telecommuter, who is able to run very fast, is running horizontally to the right at a speed $v = \sqrt{3}v_r$. At what angle from the vertical does the telecommuter see the rain falling?

(A) 0° (straight down)
(B) 30°
(C) 45°
(D) 60°
(E) 90°
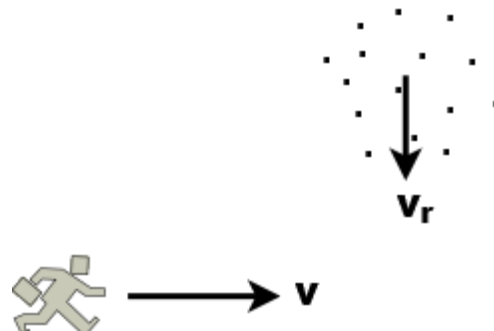
18. Suppose you try to head straight across the river of problem 15, rowing at 3 km/hr as you would in still water. How long does it take you to cross the river compared to the time it would take you in still water?

    (A) Cannot be determined from the information given
    (B) You cannot reach the other bank in a finite amount of time.
    (C) Less time
    (D) More time
    (E) The same amount of time

19. You can row a boat in still water at a speed of c km/hr. You decide to make a round trip in your boat down a river a distance L and back. The river is flowing to the right at a speed $v < c$, as in the figure below.
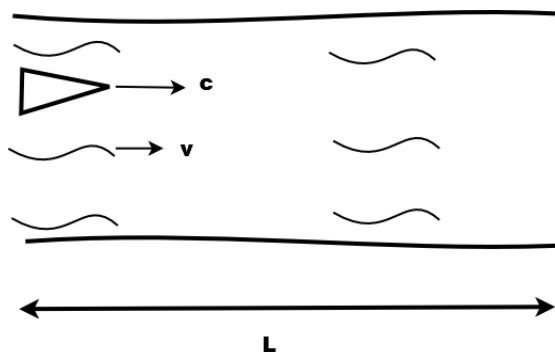
The time it takes you to make the round trip is:

    (A) L/c
    (B) $2L/c$
    (C) Less than $2L/c$
    (D) Greater than $2L/c$
    (E) Cannot be determined from the information given

20. A ball is thrown off a building horizontally with a velocity of 10 m/s. The building is 45 m tall. Ignoring air resistance, the ball lands a horizontal distance from the building of approximately:

    (A) 90 m
    (B) 30 m
    (C) 20 m
    (D) 10 m
    (E) 3 m

21. A raindrop has been falling downward through the air long enough so that it has reached a constant terminal velocity of 40 m/s. Over the next 3 seconds it will travel a further distance of:

    (A) 120 m
    (B) 180 m
    (C) 240 m
    (D) 320 m
    (E) 400 m

22. Two balls are pushed off a table of height h. Ball a has negligible horizontal velocity and falls straight down. Ball b has an initial horizontal velocity v_{ox}. Neglecting air resistance:

    (A) ball a hits the ground first.
    (B) ball b hits the ground first.
    (C) they both hit the ground at the same time, but a's speed is greater
    (D) they both hit the ground at the same time, but b's speed is greater.
    (E) they both hit the ground at the same time with the same speed.

23. As shown in the figure below, two balls are shot at initial velocity v_o along two tracks of horizontal length L and a height h above the floor. The first track is straight. After a length L_1, track b dips to the ground for a length L_2, then rises back to its original height for a length L_3.
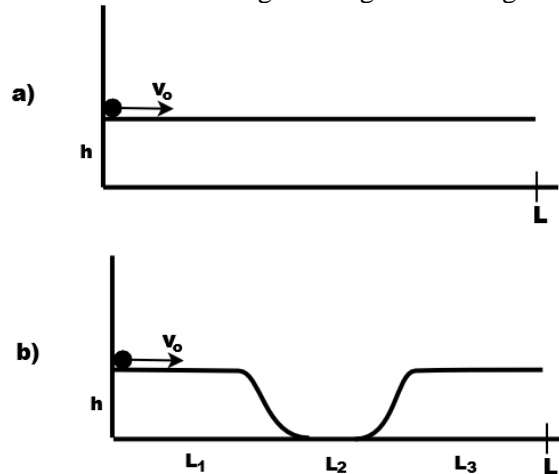
Ignoring any friction and the small curved sections of the track, which ball reaches the end of its track first?

    (A) Ball a
    (B) Ball b
    (C) They both reach the ends of their tracks at the same time with the same speed.
    (D) They both reach the ends of their tracks at the same time with different speeds.
    (E) The answer depends on the height h.

24. The conductor on a train spots a cow on the rails 200 meters ahead and immediately applies the brakes. The train decelerates at a constant deceleration a until it stops after 20 seconds, just before hitting the cow. Can the initial speed of the train in the instant before the brakes were applied be determined?

    (A) Yes, by dividing the given time by the given distance.
    (B) No, because to find the initial speed requires the deceleration, and the deceleration cannot be determined.
    (C) Yes, by solving for a in the equation $2as = v_o^2 - v_f^2$ and then using the equation $s = -\left(\frac{1}{2}\right)at^2 + v_ot$.
    (D) Yes, by finding the displacement in terms of the average speed.
    (E) Both (C) and (D).

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About the Author
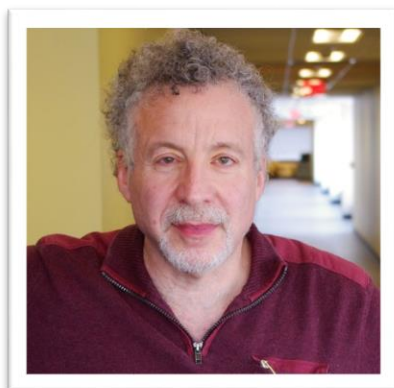
Tony Rothman is a theoretical cosmologist who holds a PhD in physics from the University of Texas, Austin. In 2016 he joined the faculty of NYU's Tandon School of Engineering, having previously taught at Princeton and Harvard Universities. To date Rothman has authored approximately 60 scientific papers as well as 11 previous books, both fiction and nonfiction. The latest of these are two novels, *Firebird*, about a race for nuclear fusion; and *The Course of Fortune*, a historical novel about the Great Siege of Malta in 1565. His book *Sacred Mathematics, Japanese Temple Geometry*, with Hidetoshi Fukagawa, won the 2008 American Association of Publishers Award for Professional and Scholarly Excellence in mathematics. He has won numerous other writing awards and has been nominated for the Pulitzer Prize.

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Dr. Steve Warner earned his Ph.D. at Rutgers University in Mathematics, and he currently works as an Associate Professor at Hofstra University. Dr. Warner has more than 20 years of experience in general math tutoring and more than 15 years of experience in SAT math tutoring. He has tutored students both individually and in group settings and has published several math prep books for the SAT, ACT and AP Calculus exams.

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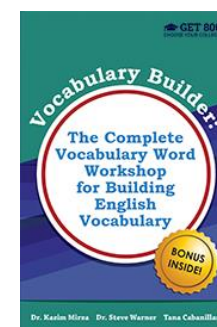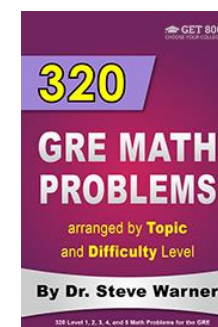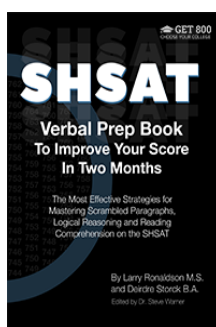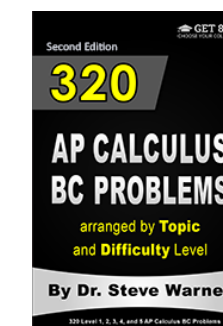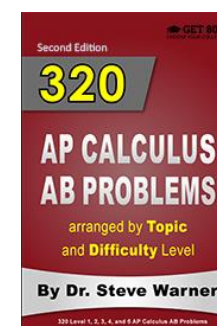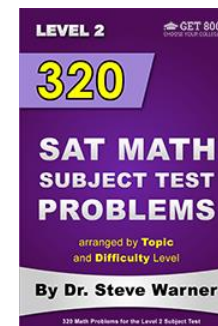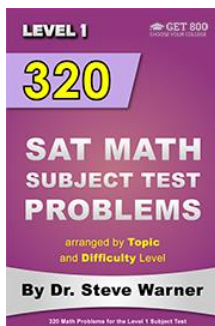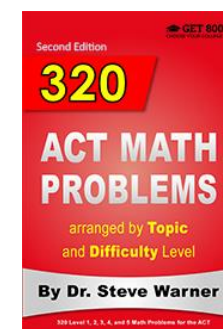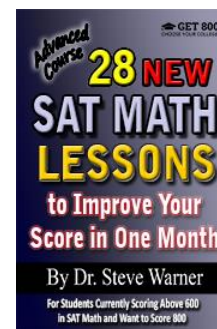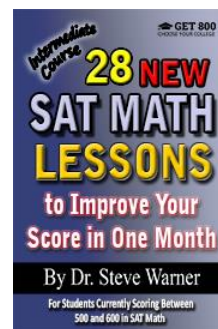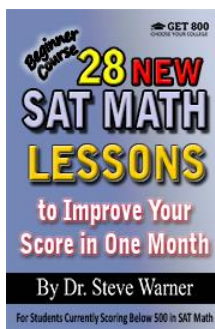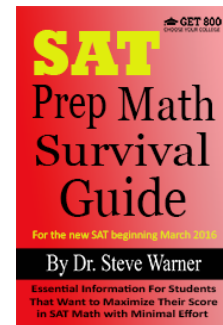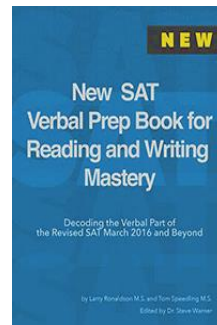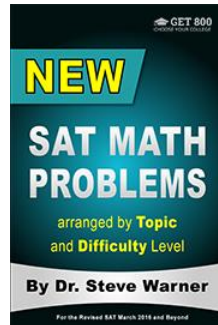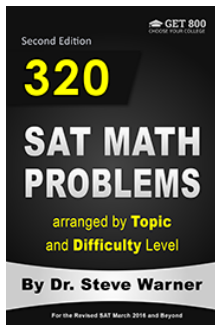
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